Intermediating Auctioneers

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Abstract

Auction theory almost exclusively assumes that the auctioneer and the owner (or the buyer) of the goods are one and the same. In reality, however, most auctioneers act as intermediaries between buyers and sellers. Despite the fast growth of these intermediating auctioneers in recent years, essentially no studies exist on how they set commissions. In this paper, we make a first step towards filling in the gap by studying a model with $J$ auctioneers, $K$ sellers, $N$ buyers, where $J \leq K \leq 2$, and each seller has an item to sell in a standard auction, where buyers have private and independent values. Sellers are located on the two ends of the unit interval. They have correlated values drawn from a power distribution. Observing the percentage fees set by the auctioneers, each seller chooses an auctioneer to sell her item and sets a reserve price. Observing the choices of the sellers, each buyer chooses at most one item to bid for. We explicitly model the competition between auctioneers with and without capacity constraints. We show that when discriminatory pricing and rationing are not allowed, equilibrium percentage fees are independent of the number of buyers and the distribution of buyer’s values. In the monopoly auctioneer case and the capacity constraint case with non-discriminatory rationing, equilibrium percentage fees depend only on the parameter of the power distribution. In the capacity constraint case with discriminatory rationing, auctioneers play mixed strategies in equilibrium, and percentage fees are lower. Absent capacity constraints, equilibrium percentage fees are zero due to the Bertrand-type competition.

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PREFACE

Thesis Title: Three Essays in Voting, Social Choice and Intermediation
Supervisors: Peter Bardsley, Simon Loertscher, Andrew McLennan

This thesis consists of three independent papers in the related fields of social choice theory, political economy, and auction theory. The paper Intermediating Auctioneers will be presented at the conference. The extended abstracts of the other two papers are as follows.

1. Poisson Games, Strategic Candidacy and Duverger’s Law

Abstract. Duverger’s law predicts a long-run two-candidate stable outcome under a plurality voting system. Duverger (1954) explains the law using the waste vote argument, which emphasizes voters’ tendency to abandon candidates commonly perceived to have the least support. Palfrey (1989) formalizes Duverger’s argument by modeling a three-candidate voting situation as a Bayesian game, and shows that there exists a symmetric Bayesian equilibrium where the vote share of the weakest candidate is asymptotically zero. However, Palfrey’s model does not apply to non-generic cases where two weaker candidates have the same expected vote share. In this paper, we use strategic candidacy and the waste vote argument to fully explain Duverger’s law. We add uncertainty about the number of voters to Palfrey’s framework, and model the election situation as a two-stage game where candidates make strategic entry decisions in the first stage, followed by plurality voting, which is modeled as a Poisson game introduced in Myerson (1998, 2000). Using our framework, we successfully explain both generic and non-generic cases in Palfrey (1989), as well as an anomaly to Duverger’s law, viz. in India there persists a strong central party and two weaker parties with similar strength.

2. Monotonicity and Candidate Stable Voting Correspondences

Abstract. Dutta, Jackson and Le Breton (Econometrica, 2001) initiated the study of strategic candidacy. A voting procedure satisfies candidate stability if no candidate has incentives to withdraw her candidacy in order to manipulate the voting outcome in her favor. Dutta et al. show that a single valued voting procedure satisfying candidate stability and unanimity must be dictatorial if voters have strict preferences and candidates cannot vote. Eraslan and McLennan (JET, 2004) extend this result to a framework that allows weak preferences and multi-valued voting procedures (voting correspondences). They obtain the existence of a serial dictatorship under a stronger version of candidate stability. We show that voting correspondences satisfying strong candidate stability and unanimity are monotonic, that is, if a winning candidate’s position is weakly improved in all voters’ preference rankings, then the candidate remains a winner. Monotonicity provides a direct link between the standard dictatorship in Dutta et al. and the serial dictatorship in Eraslan and McLennan. Using this particular property of voting correspondences, we provide an alternative proof to the Eraslan and McLennan’s result.
1 Introduction

Auction theory almost exclusively assumes that the auctioneer and the owner of the goods (or the buyer in the case of procurement auction) are one and the same. In reality, however, most auctions are run by auctioneers who do not own nor buy the goods to be sold. Therefore, these auctioneers act as intermediaries between buyers and sellers. Examples include internet auction sites like eBay, auction houses like Christie’s or Sotheby’s, and Australian real estate brokers. These intermediating auctioneers have been experiencing fast growth in recent years. However, despite the burgeoning research on intermediaries as two-sided market platforms, essentially no studies exist on how intermediating auctioneers set their commission fees. Understanding the behavior of these auctioneers is obviously important. Consider, for example, the allegations that Christie’s and Sotheby’s were collusively setting their commission fees above the competitive level. Clearly, such a statement presumes that collusive fees are higher than competitive fees, which is basically a translation of insights from models of price setting to models of commission fee setting. However, it is not clear a priori whether the intuition from the former carries over to the latter, and we shall argue that indeed it does not. That is, we will show that there are very plausible assumptions under which the collusive commission fees are exactly the same as the Nash equilibrium fees.

In this paper, we make a first step towards building a theory of intermediating auctioneers. Our contributions are twofold. We introduce a model of intermediating auctioneers, and we provide a framework to analyze competition between these auctioneers. Intermediating auctioneers typically charge sellers and/or buyers percentage fees on the sales price. 

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1In 2006, Sotheby’s auction sales totaled a historical record of $3.75 billion, a 36% increase over 2005 sales. Christie’s art auction sales totaled $4.414 billion, also a 36% increase. At the online auction sites eBay, registered users increased by 23% over 2005 to reach 222 million (active users 82 million) at the end of 2006, generating a total of $2.365 billion listings. Also see Lucking-Reiley (2000).


3To the best of our knowledge, the only paper that deals with this topic is Ginsburgh, Legros and Sahuguet (2005). However, they only study the welfare effects of an exogenous increase in commissions on buyers and sellers. In particular, they do not address the auctioneer’s profit maximization problem.

4For example, Christie’s charges a 20% commission on the first $100,000 and 12% thereafter. Sotheby’s charges 20% on the first $200,000 and 12% on anything above. In addition, both auction houses charge buyer’s premium, which typically starting from 25% at Christie’s and 20% at Sotheby’s. eBay charges a listing fee ranged from $0.20 to $4.80, and 5.25% of the initial sales price $0.01-$25, plus 3.25% of the next $25.01-$1,000, plus 1.5% of the remaining balance. Amazon charges a $0.99 listing fee, and 6% of the sales price for computers, 8% for camera & photo and electronics, 10% for items in the Everything Else Store, and 15% for all other product lines. Yahoo offers free services for using its auction websites (Yahoo US and
Therefore, it is natural to assume that auctioneers compete in percentage fees. Specifically, we study the following three stage game with $J$ auctioneers, $K$ sellers, and $N$ buyers, where $J \leq K \leq 2$. Each seller has an item to sell in a standard auction with independent and private values. Sellers are located on the two ends of the unit interval, with identical values $c$ drawn from a power distribution $H(c) = c^\gamma$, where $\gamma$ is a positive constant (see e.g. Hörner and Sahuguet (2007)). In each stage, players’ actions are taken simultaneously. In the first stage, each auctioneer sets a non-discriminatory percentage fee, i.e. it is the same for both sellers if there are two. In the second stage, each seller observes the fees. If the seller decides to sell her item, she chooses an auctioneer and sets a reserve price. In the last stage, each buyer observes the choices of the sellers, and chooses at most one auction to attend. All participating buyers use the same bidding strategies.

This setup is general enough to incorporate both cases of monopoly auctioneer and competing auctioneers. For $J = 1$ and $K = 1$, our model is reduced to a standard auction setting with a broker. For $K = 2$, we have a Hotelling auction when two differentiated products on 0 and 1 are auctioned off, paralleling the Hotelling-Bertrand model on price competition. The Hotelling auction is conducted by a monopoly auctioneer ($J = 1$) or two competing auctioneers ($J = 2$). In the latter case, two further subcases occur, where auctioneers may or may not face binding capacity constraints. We call these two subcases Bertrand-Edgeworth competition and Bertrand competition between auctioneers, respectively. When auctioneers are capacity constrained, rationing rules matter. In this paper, we focus on the study of competing auctioneers, which has the two monopoly cases as its special applications. Both competition cases are quite commonly observed in reality. Christie’s and Sotheby’s can be considered as an example of Bertrand-Edgeworth competition between auctioneers, whereas eBay and Amazon can fit into Bertrand-type competition. Bertrand-Edgeworth competition is particularly relevant for traditional brick and mortar auctioneers. For example, it

Canada auctions sites were retired on June 16, 2007). Australian real estate brokerage commissions typically range from 2.5% to 5%, plus administration fees for advertising, etc.

5 For conciseness, we will always use plural form when describing the setup. It should be understood from the context that we refer to singular form when there is only one auctioneer and/or one seller.

6 In models of product market competition with price setting sellers, equilibrium outcomes depend on rationing rules (see Kreps and Scheinkman (1983), Davidson and Deneckere (1986)). The same applies here.

7 See Loertscher (2007) for a study on the price competition among capacity constrained intermediaries. Our results on competing auctioneers are in line with his findings that competition is substantially softened in the presence of (costly) capacity constraints.
takes considerably amount of preparations for auctioning a house, including property evaluations/appraisals, advertising, pre-sale inspections, etc. Consequently, even a big real estate agent usually holds a limited number of auctions at a time. The capacity constraints are even more prominent when there is some requirement of specialist knowledge on the auctioneer’s behalf such as in the art auctions.

We obtain the following results. In the monopoly auctioneer case and Bertrand-Edgeworth competition with non-discriminatory rationing, equilibrium percentage fees depend only on the parameter \( \gamma \) of seller’s value distributions. Specifically, each auctioneer sets the same percentage fee equal to \( \frac{1}{\gamma+1} \). In Bertrand-Edgeworth competition with discriminatory rationing, auctioneers play the same mixed strategy in equilibrium, and the upper bound of the support of the mixed strategy is equal to \( \frac{1}{\gamma+1} \). Absent capacity constraints, equilibrium percentage fees are driven to zero due to Bertrand-type competition between auctioneers. The Hotelling auction used in our model is a very general form of modelling competition between sellers to auction horizontally differentiated products. Thus, our results are readily applied to a wide range of product differentiation cases. Our results on different rationing rules imposed on capacity constrained auctioneers have interesting policy implications as it shows that equilibrium percentage fees are lower under discriminatory rationing rule.

There are several papers modelling competition among sellers on reserve prices. McAfee (1993), Peters (1997), Peters and Severinov (1997, 2006) all assume a large number of sellers in a perfectly competitive market setting. Burguet and Sákovics (1999) analyzes competition between two owners of homogeneous goods. The paper most closely related to the Hotelling auction in our model is Parlane (2005). However, in her model sellers’ values are assumed to be zero. Assuming positive values in our model allows us to evaluate the effect of percentage fees on sellers’ equilibrium reserve prices. The only paper that studies percentage fees set by intermediaries is Loertscher and Niedermayer (2007). They show that in the case of one buyer, one seller and one monopolistic intermediary, percentage fees levied on the price set by the seller implement the optimal mechanism that maximizes the intermediary’s expected profit if and only if the seller’s values are drawn from a generalized power distribution. This provides a rationale for the use of power distributions in our model.

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8Surprisingly, the study on Hotelling auction is almost non-existent in the auction theory literature. To the best of our knowledge, the only paper that studies a Hotelling auction is Parlane (2005).
The remainder of the paper is structured as follows. Section 2 describes both the general setting and specific applications of the model. Second 3 briefly summarizes the Hotelling-Bertrand Model, which closely parallels the Hotelling auction setting. Section 4 studies Bertrand-Edgeworth competition between auctioneers, and applies the results to both cases of monopoly auctioneer. Section 5 considers Bertrand competition. Section 6 concludes.

2 The Model

We first lay out the general model and then briefly discuss four important special cases.

2.1 The general setting

There are $K \leq 2$ sellers. Each seller has an item to auction and values her item at $c$. We assume that $c$ is distributed on $[0, 1]$ according to the distribution function $H$ given by $H(c) = c^\gamma$, where $\gamma$ is some positive constant, and $h(c) = \gamma c^{\gamma-1}$ is the associated density function. Notice that the uniform distribution is a special case with $\gamma = 1$. Sellers are located on the two ends of the interval $[0, 1]$. We refer to the seller at 0 as $k = 1$ and the seller at 1 as $k = 2$. The auctions are conducted by $J \leq K$ auctioneers. Auctioneer $j$ charges the sellers served by him a percentage fee $\tau_j \in [0, 1]$ on the sales price. That is, we assume that auctioneers cannot price discriminate.

There are $N > 1$ buyers, each bidding for at most one item. Each buyer has a type $\theta \in [0, 1]$, which is independently and identically drawn from a continuous distribution function $F$ with density $f$. A type $\theta$ buyer’s valuation for seller $k$’s item is $v_k(\theta)$. We denote the induced cumulative distribution function of $v_k(\theta)$ by $\Phi_k$ with density $\phi_k$, and define the hazard rate to be $\lambda_k(v_k) = \frac{\phi_k(v_k)}{1-\Phi_k(v_k)}$. The respective valuations of a buyer of type $\theta$ for the two items are\(^\dagger\)

\[ v_1(\theta) = 1 - t\theta \text{ and } v_2(\theta) = 1 - t(1 - \theta), \]

where $t \in (0, 1]$ describes the degree of product differentiation. Notice when $t = 0$, $v_1(\theta) = v_2(\theta) = 1$, for all $\theta \in [0, 1]$, that is, $\Phi_k$ would be a degenerated distribution. To obtain

\(^\dagger\)We refer to independent values as each buyer’s type being independently distributed according to $F$, but notice the valuations of a buyer of type $\theta$ for the two items are correlated.
interesting results, we thus assume \( t > 0 \). \( \Phi_k \) and \( \phi_k \) satisfy the following conditions$^{10}$:

\[
F(\theta) = 1 - \Phi_1(v_1(\theta)) = \Phi_2(v_2(\theta)) \quad \text{and} \quad f(\theta) = t\phi_1(v_1(\theta)) = t\phi_2(v_2(\theta)).
\]

We call this type of auctions for two items at 0 and 1 the Hotelling auction because of its close relation to the Hotelling-Bertrand model on price competition, which is briefly discussed in the next section.

We make standard regularity assumptions on buyers’ type distributions.

**Assumption 1.** For each \( \theta \in [0, 1] \), \( f(\theta) > 0 \), and \( \frac{f(\theta)}{F(\theta)} \) is strictly decreasing in \( \theta \), \( \frac{f(\theta)}{1-F(\theta)} \) is strictly increasing in \( \theta \).

Notice Assumption 1 implies that for each \( \theta \in [0, 1] \), \( \theta + \frac{F(\theta)}{f(\theta)} \) and \( \theta - \frac{1-F(\theta)}{f(\theta)} \) are strictly increasing in \( \theta \). We call this increasing virtual valuations.

The realizations of \( c \) and \( \theta \) are private information of sellers and buyers, respectively. \( H \) and \( F \) are common knowledge among all players$^{11}$. The auction format for each item is second price sealed bid auction$^{12}$.

### The Game

The sequence of the game is as follows. At each stage of the game, players’ actions are taken simultaneously.

**Stage 1.** Each auctioneer \( j \) announces his percentage fee \( \tau_j \). Let \( \bm{\tau} = (\tau_1, \tau_2) \) denote the vector of percentage fees charged by the two auctioneers if \( J = 2 \).

**Stage 2.** Each seller \( k \) observes the percentage fees and her value \( c \). If she decides to sell her item, she chooses one auctioneer to attend if \( J = 2 \). Otherwise, she just decides to be active or inactive. If both sellers attend the same auctioneer who is capacity constrained, a rationing rule is used to determine which seller gets served. We consider two specific rules. The discriminatory rationing rule lets the auctioneer

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$^{10}$If \( x \) is a random variable with density \( f(x) \), and \( y = g(x) \), where \( g \) is monotonic, then \( y \) has the density given by \( \left| \frac{1}{g'(g^{-1}(y))} \right| f(g^{-1}(y)) \), where \( g^{-1} \) is the inverse function, and \( g' \) is the derivative.

$^{11}$Assuming common knowledge of \( F \) among all players is stronger than necessary for our results. In fact, except in Bertrand-Edgeworth competition with discriminatory rationing, auctioneers need not know the distribution of \( F \).

$^{12}$Our results go through to any standard auction that satisfies the revenue equivalence principle.
choose a seller, and the non-discriminatory rationing rule assigns each seller to the auctioneer with equal probability. A seller with value \( c \) obtains a utility of \( c \) if she is inactive. If seller \( k \) is served by an auctioneer, she announces her reserve price \( r_k \).

Stage 3. Each buyer observes which auctioneer hosts which seller, the reserve prices \( r_k \), and his own type \( \theta \), and then decides whether to participate, which auction to attend if there are two, and what bidding strategies to use if he participates. A buyer obtains a utility of 0 if he does not attend any auction. Each buyer observes the number of other buyers in the auction he attends.

### 2.2 Applications of the general model

The above model incorporates the following four situations as special cases.

(i) Monopoly auctioneer with one seller.

The general model is reduced to a standard auction setting with the exception that the auctioneer is an intermediary who charges the seller a percentage fee. A type \( \theta \) buyer’s valuation for the seller’s item is either \( v_1(\theta) = 1 - t\theta \) or \( v_2(\theta) = 1 - t(1 - \theta) \), depending on whether the seller is located at 0 or 1. \( v_k \) is distributed on \([1 - t, 1]\) according to \( \Phi_k \) derived from \( F \) as follows. For \( v_k \in [1 - t, 1] \),

\[
\Phi_1(v_1) = 1 - F\left(\frac{1 - v_1}{t}\right) \quad \text{and} \quad \Phi_2(v_2) = F\left(\frac{v_2 - (1 - t)}{t}\right).
\]

We can check that the property of increasing virtual valuations carries over from \( F \) to \( \Phi_k \). That is, for each \( v_k \in [1 - t, 1] \), \( v_k - \frac{1 - \Phi_k(v_k)}{\sigma_k(v_k)} (= \Upsilon_k) \) is strictly increasing in \( v_k \).\(^{13}\)

For the rest of the paper, when \( K = 1 \), we let the seller be located at 1, and drop the subscript \( k \) whenever applicable, with the understanding that \( k = 2 \). Notice when \( t = 1 \), we have \( v_2(\theta) = \theta \), so this case would coincide with the standard auction setting with the extra element of an intermediary.

\(^{13}\)To see this, notice \( \theta = \frac{1 - v_1}{t} = \frac{v_2 - 1 + t}{t} \), and \( \Upsilon_1 = 1 - t \left( \theta + \frac{F(\theta)}{f(\theta)} \right) \), \( \Upsilon_2 = 1 - t \left( 1 - \left( \theta - \frac{1 - F(\theta)}{f(\theta)} \right) \right) \). So 
\[
\frac{\partial \Upsilon_1}{\partial v_1} = \frac{\partial}{\partial \theta} \left( \theta + \frac{F(\theta)}{f(\theta)} \right) > 0, \quad \text{and} \quad \frac{\partial \Upsilon_2}{\partial v_2} = \frac{\partial}{\partial \theta} \left( \theta - \frac{1 - F(\theta)}{f(\theta)} \right) > 0.
\]
(ii) Monopoly auctioneer with competing sellers.

There are two sellers $k = 1, 2$ located on the two ends of the interval $[0, 1]$. This is the case described in the general setting with $J = 1$. Sometimes we refer to the two sellers as $k$ and $-k$ when there is no need to specify their positions.

(iii) Competing auctioneers with capacity constraints and competing sellers.

There are two auctioneers $j = 1, 2$, each with the capacity of running one auction. If an auctioneer attracts both sellers, a pre-specified rationing rule is applied. We study both cases when the rule is discriminatory vs non-discriminatory. The seller who is rejected can go to the other auctioneer or stay out. A seller randomizes with equal probability between the two auctioneers if she is indifferent.

(iv) Competing auctioneers without capacity constraints and competing sellers.

This is similar to case (iii) except that each auctioneer has the capacity of running two auctions, therefore no rationing rule needs to be specified.

Finally, we briefly comment on the Hotelling auction used in our model. Hotelling auctions are quite common in practice. For example, in Australia, land and houses are usually sold in auctions. Typically, there are several real estate agents, i.e. auctioneers, who compete on commission fees to attract sellers. Each seller usually has one property to sell, and buyers are physically constrained to attend one auction at a time when several auctions are simultaneously conducted at different locations. Indeed, any two (or more) auctions that sell horizontally differentiated products, when each buyer is physically or financially constrained to buy only one unit of the good, can be considered as some variant of a Hotelling auction.

3 Preliminary: The Hotelling-Bertrand Model

Before analyzing buyers’ and sellers’ equilibrium behavior, we briefly summarize the Hotelling-Bertrand model (also known as the differentiated products Bertrand model) because our Hotelling auction setting closely parallels it, where reserve prices instead of prices are the strategic variables of the sellers. In the Hotelling-Bertrand model, there are two sellers
\( k = 1, 2 \) located on 0 and 1. They have the same marginal cost \( c \in [0, 1] \), and simultaneously set prices \( p_k \) to maximize their expected profits. There is a continuum of buyers uniformly distributed on \([0, 1]\), each demanding one unit of the good. A buyer assigns the same value 1 to each item, and pays the price for the item purchased plus a transportation cost \( t \in [0, 1] \) per unit distance, where \( t \) can be interpreted as a parameter for product differentiation.

As a function of \( c \) and \( t \), there are three regions, each giving rise to a different type of equilibrium. The regions are illustrated in Figure 1.

![Figure 1: Three regions of equilibria depending on \( t \) and \( c \)](image)

**Region 1. \( t > 1 - c \): Local monopolist sellers.**

When sellers’ marginal costs are large, and/or products are highly differentiated, sellers are local monopolists in their own markets. That is, for \( t > 1 - c \), and \( c \in (0, 1] \), there exist two marginal buyers \( \hat{\theta}_1, \hat{\theta}_2 \) with \( \hat{\theta}_1 < \hat{\theta}_2 \) such that all \( \theta \in [0, \hat{\theta}_1] \) buy from seller 1, all \( \theta \in [\hat{\theta}_2, 1] \) buy from seller 2, and all \( \theta \in (\hat{\theta}_1, \hat{\theta}_2) \) do not buy any item. Buyers \( \hat{\theta}_1, \hat{\theta}_2 \) get zero surplus. The equilibrium prices are \( p_1 = p_2 = \frac{1 + c}{2} \), and equilibrium profits are \( \Pi_1 = \Pi_2 = \frac{(1-c)^2}{4t} \).

\(^{14}\)When \( t = 1 - c \), and \( c \in [0, 1] \), \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) converge to \( \tilde{\theta} \), where \( \tilde{\theta} \) is the buyer who is indifferent between...
Region 2. $t \leq \frac{2}{3}(1 - c)$: Duopolist sellers.

When $t$ and $c$ are low such that $t \leq \frac{2}{3}(1 - c)$ and $c \in [0, 1]$, equilibrium prices will be low and there exists a unique indifferent buyer $\bar{\theta}$ who gets positive surplus in equilibrium. All $\theta \in [0, \bar{\theta})$ buy from seller 1, and all $\theta \in (\bar{\theta}, 1]$ buy from seller 2. The equilibrium prices are $p_1 = p_2 = t + c$, and equilibrium profits are $\Pi_1 = \Pi_2 = \frac{t}{2}$.

Region 3. $\frac{2}{3}(1 - c) < t < 1 - c$: Constrained duopolist sellers\(^{15}\).

When $t$ and $c$ are in the intermediate range, i.e. $\frac{2}{3}(1 - c) < t < 1 - c$ and $c \in [0, 1)$, there are multiple equilibria. In each equilibrium, there is an indifferent buyer $\hat{\theta}$ who gets zero surplus. That is, $p_1$ and $p_2$ are such that $1 - p_1 - t\hat{\theta} = 1 - p_2 - t(1 - \hat{\theta}) = 0$. So equilibrium prices satisfy $\hat{p}_1 + \hat{p}_2 = 2 - t$, $p_1, p_2 \in [1 - t, 1]$. Seller 1 gets all $\theta \in [0, \hat{\theta})$, and seller 2 gets all $\theta \in (\hat{\theta}, 1]$. Figure 2 illustrates sellers’ best response functions $BR_k$.

Any point on the bold line corresponds to an equilibrium for this constrained duopoly case.

\[ BR_1(p_2) = BR_2(p_1) \]

Figure 2: Multiple equilibria in the constrained duopoly case

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\(^{15}\)This is perhaps the least known type of equilibria, but it obtains generically in this type of model, see e.g. Salop (1979). In particular, it cannot be avoided if, as we assume, $c$ is distributed continuously on $[0, 1]$. The standard approach is to focus on the unique symmetric equilibrium, see e.g. Chen and Riordan (2007).
4 Bertrand-Edgeworth Model

In this section, we analyze the full game with two capacity constrained auctioneers. As in the Hotelling-Bertrand model, let $\tilde{\theta}_k$ be the type who is indifferent between attending auction $k$ and not participating when in equilibrium sellers are local monopolists. Here $auction k$ refers to the auction for seller $k$’s item. Let $\tilde{\theta}$ be the type who is indifferent between attending auction $k$ and $-k$ when in equilibrium sellers are duopolists, and $\hat{\theta}$ be the indifferent type when sellers are constrained duopolists.

4.1 The buyers’ subgame

Each buyer chooses a participation strategy $not participating$, $attending auction k$, or $attending auction -k$, and a bidding strategy $b$ to maximize his expected utility, where $b : [0,1] \to \mathbb{R}^+$ is a mapping from the type space to the set of bids. We assume that buyers follow the same bidding strategy. As is well known, it is a weakly dominant strategy for each buyer to bid his true valuation in second price auctions. That is, a buyer’s bidding strategy is $b(\theta) = v_k(\theta)$ if he attends auction $k$. As is standard, we focus on the equilibrium where buyers play weakly dominant strategies, and do not further mention this restriction.

Buyers’ participating strategies depend on sellers’ reserve prices and the degree of product differentiation. If $t$ is high, buyers are completely separated in two submarkets. That is, there is no buyer who expects non-negative utility from both auctions if he bids truthfully. The condition and the characterization of this equilibrium is given in the following proposition.

Proposition 4.1. Suppose that reserve prices and $t$ are such that $t > 2 - r_1 - r_2$. Then there exists a unique equilibrium in the buyers’ subgame where all buyers with $\theta \leq \tilde{\theta}_1 = \frac{1 - r_1}{t}$ attend auction 1, all buyers with $\theta \geq \tilde{\theta}_2 = \frac{r_2 - 1 + t}{t}$ attend auction 2, and all buyers with $\theta \in (\tilde{\theta}_1, \tilde{\theta}_2)$ do not attend any auction.

Proof. Since buyers bid truthfully in equilibrium, they attend auction $k$ if and only if their valuations for item $k$ are no less than the reserve price $r_k$. Note that $r_k = v_k(\tilde{\theta}_k)$, $k = 1, 2$, and $t > 2 - r_1 - r_2 \iff \tilde{\theta}_1 < \tilde{\theta}_2$. For all $\theta \leq \tilde{\theta}_1$, $v_1(\theta) \geq r_1$, $v_2(\theta) < r_2$. So all these buyers attend auction 1. Similarly, all buyers with $\theta \geq \tilde{\theta}_2$ attend auction 2. For all $\theta \in (\tilde{\theta}_1, \tilde{\theta}_2)$,
\[ v_k(\theta) < r_k, \ k = 1, 2, \ \text{so these buyers do not participate.} \]

When \( t = 2 - r_1 - r_2 \), the equilibrium in Proposition 4.1 remains true with the slight modification that \( \tilde{\theta}_1 = \tilde{\theta}_2 \), and this is the unique type indifferent between attending the two auctions.

When products are less differentiated and/or reserve prices are lower, sellers’ markets overlap in the sense that some buyer’s valuation for either item is higher than the associated reserve price. This buyer attends the auction that gives him higher expected utility. In particular, there exists some indifferent type who obtains the same expected utility from attending either auction. This result is given in the following proposition.

**Proposition 4.2.** Suppose that reserve prices and \( t \) are such that \( t < 2 - r_1 - r_2 \). Then there exists a unique \( \bar{\theta} \in (\frac{r_2-1+t}{t}, \frac{1-r_1}{t}) \) such that in equilibrium, all buyers with \( \theta < \bar{\theta} \) attend auction 1, all buyers with \( \theta > \bar{\theta} \) attend auction 2, and \( \bar{\theta} \) is indifferent between attending the two auctions, where \( \bar{\theta} \) is such that

\[
\Phi_1^{N-1}(v_1(\bar{\theta}))(v_1(\bar{\theta}) - r_1) = \Phi_2^{N-1}(v_2(\bar{\theta}))(v_2(\bar{\theta}) - r_2). \tag{1}
\]

**Proof.** Let \( \bar{\theta}_1 = \frac{1-r_1}{t} \), \( \bar{\theta}_2 = \frac{r_2-1+t}{t} \). Then \( r_k = v_k(\bar{\theta}_k) \), \( k = 1, 2 \), and \( t < 2 - r_1 - r_2 \iff \bar{\theta}_2 < \bar{\theta}_1 \). Define the function \( \Delta : [\bar{\theta}_2, \bar{\theta}_1] \to \mathbb{R} \) by

\[
\Delta(\theta) = \Phi_1^{N-1}(v_1(\theta))(v_1(\theta) - v_1(\bar{\theta}_1)) - \Phi_2^{N-1}(v_2(\theta))(v_2(\theta) - v_2(\bar{\theta}_2)),
\]

which measures the differences in a buyer’s expected surplus from attending the two auctions. It is continuous and decreasing in \( \theta \) on its domain, and \( \Delta(\bar{\theta}_2) > 0 \), \( \Delta(\bar{\theta}_1) < 0 \). Therefore, there exists a unique \( \bar{\theta} \in (\bar{\theta}_2, \bar{\theta}_1) \) such that \( \Delta(\bar{\theta}) = 0 \), that is, (1) holds. So \( \bar{\theta} \) is indifferent between attending the two auctions. It is straightforward to check that all \( \theta \in [0, \bar{\theta}_2) \) attend auction 1, all \( \theta \in (\bar{\theta}_2, 1] \) attend auction 2. For all \( \theta \in [\bar{\theta}_2, \bar{\theta}_1] \), \( v_k(\theta) \geq r_k \). Furthermore, for all \( \theta \in [\bar{\theta}_2, \bar{\theta}] \), \( \Delta(\theta) > 0 \), that is, the expected surplus is higher in auction 1, therefore, all these buyers attend auction 1. For all \( \theta \in (\bar{\theta}, \bar{\theta}_1] \), \( \Delta(\theta) < 0 \), so these buyers attend auction 2. \( \square \)

### 4.2 The sellers’ subgame

Each seller first determines which auctioneer to attend if she decides to sell her item after observing the percentage fees. If seller \( k \) is hosted by some auctioneer, she sets a reserve
price $r_k$ to maximize her expected profit. In the one seller case, all buyers with valuations larger than the reserve price attend the auction. The same needs not be true in the case of competing sellers because some buyers with valuations larger than $r_k$ may still want to attend auction $-k$ since their valuations for $-k$’s item are larger. Consequently, the minimum bid in auction $k$ can be strictly larger than $r_k$. Denote by $x_k$ the minimum equilibrium bid in auction $k$, which in general depends on $r_k$ and $r_{-k}$. The problem of seller $k$ hosted by auctioneer $j$ can be written as

$$\max_{r_k} \Pi_k(c, \tau_j, r_k, r_{-k}) = (1 - \tau_j)N M_k(\Phi_k, r_k, r_{-k}) + c \Phi_k^N(x_k)$$

(2)

where $M_k(\Phi_k, r_k, r_{-k})$ is the ex ante expected payment of a buyer in auction $k$, which is calculated as follows.

$$M_k(\Phi_k, r_k, r_{-k}) = r_k(1 - \Phi_k(x_k))\Phi_k^{N-1}(x_k) + (N - 1) \int_{x_k}^1 y(1 - \Phi_k(y))\Phi_k^{N-2}(y)d\Phi_k(y)$$

(3)

The first term represents the case when there is only one buyer attending auction $k$, and he pays the reserve price $r_k$. The second term is when there are at least two buyers in auction $k$, and the winner pays the second highest bid.

Sellers’ participation strategies are as follows. First, a seller with value $c$ attends auctioneer $j$ only if $c \leq 1 - \tau_j$ since the maximum payment she can get from a buyer is 1. If both percentage fees are no greater than $1 - c$, her first choice is always the auctioneer with lower fee, since her equilibrium expected profit is decreasing in fees. Second, it is a weakly dominant strategy for the seller to go to the other auctioneer instead of staying out if she is rejected by the first. Let us now solve sellers’ equilibrium reserve prices. Notice that if an auctioneer sets percentage fee at 1, then only sellers with zero value will attend him, which is an event of zero probability. When analyzing sellers’ pricing strategies, we therefore assume that both $\tau_1$ and $\tau_2$ are strictly less than 1.

### 4.2.1 Local monopolist sellers

We have shown that buyers’ equilibrium behavior depends on the degree of product differentiation and sellers’ reserve prices. The latter, in turn, depend on sellers’ values and percentage fees, as well as on $t$ via buyers’ equilibrium bidding strategies. When products are sufficiently
differentiated, even at $c = \tau_1 = \tau_2 = 0$, equilibrium reserve prices are already high enough to separate the two markets. Each seller is a local monopolist in her own market. This intuition is formalized in Propositions 4.3.

Before we state the results, we introduce and clarify a few notations. Denote $\bar{c}$ as the highest value for a seller to be willing to attend either auctioneer for a given $\tau$. Notice $\bar{c} = 1 - \max\{\tau_1, \tau_2\}$. Denote sellers’ marginal revenues for any given $\theta$ and $t$ as

\[
MR_1(\theta, t) = 1 - t \left( \theta + \frac{F(\theta)}{f(\theta)} \right) \quad \text{and} \quad MR_2(\theta, t) = 1 - t \left( 1 - \left( \theta - \frac{1 - F(\theta)}{f(\theta)} \right) \right)
\]

In the following analysis of equilibrium reserve prices, we refer to $\tau_k$ as the percentage fee charged on seller $k$, and $c_1 - \tau_k$ as seller $k$’s effective marginal cost.

**Proposition 4.3.** There exists a unique $t^*$ such that for any $t > t^*$, $\tau_1, \tau_2 \in [0, 1)$, the following is the equilibrium outcome in the sellers and buyers’ subgame.

1. For $c < \bar{c}$, sellers set reserve prices satisfying

\[
\begin{align*}
    r_1(c, \tau, t) &= \frac{c}{1 - \tau_1} + \frac{t}{f(\tilde{\theta}_1)} F(\tilde{\theta}_1) \quad \text{and} \quad r_2(c, \tau, t) = \frac{c}{1 - \tau_2} + \frac{t}{f(\tilde{\theta}_2)} F(\tilde{\theta}_2),
\end{align*}
\]

where $\tilde{\theta}_1 = \frac{1 - \tau_1}{1 - t}$, $\tilde{\theta}_2 = \frac{1 - \tau_2}{1 + t}$.

2. All buyers with $\theta \leq \tilde{\theta}_1$ attend auction 1, all buyers with $\theta \geq \tilde{\theta}_2$ attend auction 2, and all buyers with $\theta \in (\tilde{\theta}_1, \tilde{\theta}_2)$ do not attend any auction.

At $t = t^*$, and $c = 0$, reserve prices satisfy (4) with $\tilde{\theta}_1 = \tilde{\theta}_2$.

**Proof.** We proceed in three steps. First, we show the existence and uniqueness of $t^*$ with the property stated in the proposition. Second, we show that for any given $\tau$, and $t > t^*$, the condition in Proposition 4.1 holds for $c < \bar{c}$. This then implies that buyers’ behavior stated in (2) constitutes an equilibrium. Third, we show that sellers’ reserve prices stated in (1) are equilibrium prices.

First, we show that there exists a unique $t^*$ such that $MR_k(\bar{\theta}, t^*) = 0$, for some $\bar{\theta} \in [0, 1]$. For any given $t$, we can solve $MR_1(\theta, t) = MR_2(\theta, t)$, and the solution $\bar{\theta}$ satisfies $\bar{\theta} + \frac{F(\bar{\theta})}{f(\bar{\theta})} = 1 - \left( \bar{\theta} - \frac{1 - F(\bar{\theta})}{f(\bar{\theta})} \right)$. This implies that $\bar{\theta} + \frac{F(\bar{\theta})}{f(\bar{\theta})} = \frac{1}{2} \left( 1 + \frac{1}{f(\bar{\theta})} \right) > 1$. Notice $\bar{\theta}$ is independent of $t$. Now fix $\bar{\theta}$ and consider $MR_k(\bar{\theta}, t)$ as a function of $t$. $MR_k(\bar{\theta}, t)$ is decreasing in $t$, and
\(MR_k(\tilde{\theta}, 0) = 1 > 0, MR_k(\tilde{\theta}, 1) = 1 - \left(\frac{F(\tilde{\theta})}{f(\tilde{\theta})}\right) < 0\). So there is a unique \(t^* \in (0, 1)\) such that \(MR_k(\tilde{\theta}, t^*) = 0\).

Second, we show that for any \(t > t^*, \tau_1, \tau_2 \in [0, 1]\), the condition in Proposition 4.1 holds for \(c < \tilde{c}\), i.e. reserve prices given by (4) satisfy \(t > 2 - r_1 - r_2\). Notice (4) are equivalent to

\[
MR_1(\tilde{\theta}_1, t) = \frac{c}{1 - \tau_1} \quad \text{and} \quad MR_2(\tilde{\theta}_2, t) = \frac{c}{1 - \tau_2}
\]

At \(t = t^*\), and \(c = \tau_1 = \tau_2 = 0\), \(MR_k(\tilde{\theta}, t^*) = 0\). Now for any given \(t > t^*\), and \(\tau_1, \tau_2 \in [0, 1]\), we must have \(\tilde{\theta}_1 < \tilde{\theta}, \tilde{\theta}_2 < \tilde{\theta}_2\) for (5) to hold, due to Assumption 1. So \(\tilde{\theta}_1 < \tilde{\theta}_2 \Rightarrow t > 2 - r_1 - r_2\). Buyers’ equilibrium stated in (2) follows from Proposition 4.1.

Third, we prove that reserve prices given in (4) are equilibrium prices. Since all buyers with valuations larger than or equal to \(r_k\) attend auction \(k\), we have \(x_k = r_k\). So seller \(k\)’s expected profit is

\[
\Pi_k(c, \tau_k, r_k) = (1 - \tau_k)N(M_k(\Phi_k, r_k)) + c\Phi_k^N(r_k),
\]

where

\[
M_k(\Phi_k, r_k) = r_k(1 - \Phi_k(r_k))\Phi_k^{N-1}(r_k) + (N - 1) \int_{r_k}^1 y(1 - \Phi_k(y))\Phi_k^{N-2}(y) dy.
\]

The first order condition is

\[
0 = \frac{\partial \Pi_k}{\partial \tau_k} = N\Phi_k^{N-1}(r_k)(1 - \Phi_k(r_k))((1 - \tau_k)(1 - r_k\lambda_k(r_k)) + c\lambda_k(r_k)).
\]

The optimal reserve price satisfies \(r_k - \frac{1}{\lambda_k(r_k)} = \frac{c}{1 - \tau_k}\). Rearranging, we have (4).

We show that the second order condition for a maximum is satisfied. Since \(\frac{\partial^2 \Pi_k}{\partial \tau_k^2} = t^2 \frac{\partial^2 \Pi_k}{\partial \tau_k^2}\), \(\frac{\partial^2 \Pi_k}{\partial \tau_k^2} < 0\) follows immediately from

\[
\frac{\partial \Pi_1}{\partial \tilde{\theta}_1} = N(1 - F(\tilde{\theta}_1))^{N-1}f(\tilde{\theta}_1) \left(\frac{MR_1(\tilde{\theta}_1, t) - \frac{c}{1 - \tau_1}}{1 - \tau_1}\right) = 0 \Rightarrow \frac{\partial^2 \Pi_1}{\partial \tilde{\theta}_1^2} < 0
\]

\[
\frac{\partial \Pi_2}{\partial \tilde{\theta}_2} = -NF(\tilde{\theta}_2)^{N-1}f(\tilde{\theta}_2) \left(\frac{MR_2(\tilde{\theta}_2, t) - \frac{c}{1 - \tau_2}}{1 - \tau_2}\right) = 0 \Rightarrow \frac{\partial^2 \Pi_2}{\partial \tilde{\theta}_2^2} < 0
\]

due to Assumption 1. So seller \(k\)’s expected profit obtains the maximum at \(r_k\) given by (4).

We are left to check that given monopoly reserve price \(r_2\), seller 1 has no incentive to lower her reserve price to overlap seller 2’s market. Suppose that seller 1 does deviate, then the best
she can do is to get all the buyers in the overlapping market segment. But in that case she
remains the monopolist in her own market, and the solution to her maximization problem is
given by (4). This shows that sellers’ behavior stated in (1) constitutes an equilibrium.

We call the equilibrium outcome in Proposition 4.3 the local monopoly outcome. The
following proposition shows that this remains the equilibrium outcome for high value sellers
in the case of less differentiated products.

For the rest of the paper, \( t^* \) defined as in Proposition 4.3 refers to the critical value that
separates the high and low product differentiation cases.

**Proposition 4.4.** For any given \( \tau \) and \( t < t^* \), there exists a unique \( c^{**} \) such that for
c \( (c^{**}, \bar{c}) \), the local monopoly outcome is the equilibrium outcome in the sellers and buyers’
subgame, where \( c^{**} \) is the solution to (4) for \( \tilde{\theta}_1 = \tilde{\theta}_2 \).

**Proof.** We first prove the existence and uniqueness of \( c^{**} \) that satisfies
\[
MR_1(\tilde{\theta}(c^{**}), t) - c^{**} \frac{\tau_1}{1 - \tau_1} = 0.
\]
Then, we show that for \( c \in (c^{**}, \bar{c}) \), we have \( \tilde{\theta}_1(c) < \tilde{\theta}_2(c) \), where
\( \tilde{\theta}_k(c) \) is such that \( MR_k(\tilde{\theta}_k(c), t) - c \frac{\tau_k}{1 - \tau_k} = 0 \), for \( k = 1, 2 \). This is equivalent to saying that
equilibrium reserve prices satisfy the condition in Proposition 4.1. The rest of the proof
parallels the argument in Proposition 4.3. The idea of the proof is illustrated in Figure 3.

\[
c = c^{**} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad\]

\[
c^{**} < c < \bar{c}
\]

Let \( \tau \) and \( t < t^* \) be given. First, we show that there exists a unique \( c^{**} \) such that
\[
MR_k(\tilde{\theta}, t) - c^{**} \frac{\tau_k}{1 - \tau_k} = 0 \quad \text{for some} \quad \tilde{\theta} \in [0, 1].
\]
Assume \( \tau_1 \leq \tau_2 \). (The proof for the case \( \tau_1 > \tau_2 \)
follows a very similar argument.) Let \( \mathcal{C} = (1 - \tau_1) \left( 1 - t \left( 1 + \frac{1}{f(1)} \right) \right) \).
For any \( c \in [\mathcal{C}, \bar{c}] \), define functions \( \Delta_k : [0, 1] \to \mathbb{R} \) by
\[
\Delta_k(\theta, c) = MR_k(\theta, t) - c \frac{\tau_k}{1 - \tau_k}, \quad k = 1, 2.
\]
\( \Delta_1(\theta, c) - \Delta_2(\theta, c) \). The following three steps show that for any \( c \in [\underline{c}, \overline{c}] \), there exists a unique \( \tilde{\theta} \in (0, 1] \) such that \( \Delta(\tilde{\theta}, c) = 0 \).

1. \( \Delta(\theta, c) \) is continuous and decreasing in \( \theta \).

2. At \( \theta = 0 \), \( \Delta(0, c) = t \left( 1 + \frac{1}{f(0)} \right) + \frac{c}{1-r_2} - \frac{c}{1-r_1} > 0 \).

3. At \( \theta = 1 \), \( \Delta(1, c) \leq 0 \) since \( \Delta_2(1, c) = 1 - \frac{c}{1-r_2} \geq 0 \), \( \Delta_1(1, c) = 1 - t \left( 1 + \frac{1}{f(1)} \right) - \frac{c}{1-r_1} \leq 0 \).

Now, consider \( \Delta_k(\tilde{\theta}(c), c) \) as a function of \( c \). We show that there exists a unique \( c^* \) such that \( \Delta_k(\tilde{\theta}(c^*), c^*) = 0 \), \( k = 1, 2 \). Notice any given pair \( t \) and \( \tau \) will be such that \( \underline{c} = \overline{c} \), or \( \underline{c} < \overline{c} \). In the former case, let \( c^* = 1 - \tau_2 \). In the latter case, there is a unique \( c^* \in (\underline{c}, \overline{c}) \).

This is shown in the following three steps.

1. \( \Delta_1(\tilde{\theta}(c), c) \) is decreasing in \( c \) on \( [\underline{c}, \overline{c}] \).

Taking the derivative with respect to \( c \) of the equation \( \Delta(\tilde{\theta}(c), c) = 0 \), we have

\[
\frac{\partial \tilde{\theta}(c)}{\partial c} \left( \frac{\partial MR_1(\theta, t)}{\partial \theta} - \frac{\partial MR_2(\theta, t)}{\partial \theta} \right) = \frac{1}{1-\tau_1} - \frac{1}{1-\tau_2}.
\]

Since the RHS is non-positive due to \( \tau_1 \leq \tau_2 \), and \( \frac{\partial MR_1(\theta, t)}{\partial \theta} < 0 \), \( \frac{\partial MR_2(\theta, t)}{\partial \theta} > 0 \), we have \( \frac{\partial \tilde{\theta}(c)}{\partial c} \geq 0 \). Therefore,

\[
\frac{\partial \Delta_1(\tilde{\theta}(c), c)}{\partial c} = \frac{\partial MR_1(\theta, t)}{\partial \theta} \frac{\partial \tilde{\theta}(c)}{\partial c} - \frac{1}{1-\tau_1} < 0.
\]

2. For \( \overline{c} = 1 - \tau_2 \), \( \Delta_1(\tilde{\theta}(\overline{c}), \overline{c}) = \Delta_2(\tilde{\theta}(\overline{c}), \overline{c}) = MR_2(\tilde{\theta}(\overline{c}), \overline{c}) - 1 < 0 \). The strict inequality holds since \( \underline{c} < \overline{c} \Rightarrow \Delta(1, c) < 0 \Rightarrow \tilde{\theta}(c) \in (0, 1) \).

3. For \( \underline{c} = (1-\tau_1) \left( 1 - t \left( 1 + \frac{1}{f(1)} \right) \right) \),

\[
\Delta_1(\tilde{\theta}(\underline{c})) = 1 - t \left( \tilde{\theta}(\underline{c}) + \frac{F(\tilde{\theta}(\underline{c}))}{f(\tilde{\theta}(\underline{c}))} \right) - \frac{\underline{c}}{1-\tau_1} = t \left( 1 + \frac{F(1)}{f(1)} \right) - t \left( \tilde{\theta}(\underline{c}) + \frac{F(\tilde{\theta}(\underline{c}))}{f(\tilde{\theta}(\underline{c}))} \right) > 0.
\]

Therefore, there exists a unique \( c^* \in (\underline{c}, \overline{c}) \) such that \( \Delta_1(\tilde{\theta}(c^*)) = 0 \). This implies that \( MR_k(\tilde{\theta}(c^*), t) - \frac{c^*}{1-\tau_k} = 0 \), \( k = 1, 2 \).

Second, we show that reserve prices given by (4) satisfy \( t > 2-r_1-r_2 \) for \( c \in (c^*, \overline{c}) \). To see
this, note that for \((4)\) to hold at \(c \in (c^*, \bar{c})\), we must have \(\tilde{\theta}_1(c) < \tilde{\theta}(c^*)\) and \(\tilde{\theta}(c^*) < \tilde{\theta}_2(c)\), due to Assumption 1. Therefore, \(\tilde{\theta}_1(c) < \tilde{\theta}_2(c) \Rightarrow t > 2 - r_1 - r_2\).

The rest of the proof is the same as in Proposition 4.3.

The preceding two propositions consider cases where the realizations of sellers’ values are such that both prefer attending some auctioneer to staying out. The following proposition characterizes the situation where one auctioneer’s percentage fee is too high at a particular realization of \(c\) to attract either seller. In this case, both sellers first attend the auctioneer with the lower fee, and the one who gets rejected stays out. The seller in the auction is the monopolist in the whole market, and her equilibrium behavior remains the same as in the local monopoly case.

**Proposition 4.5.** Suppose that \(1 - \tau_1 - j \leq c < 1 - \tau_j\). Then in equilibrium both sellers go to auctioneer \(j\), and the one who is rejected stays out. If seller \(k\) gets served, she sets reserve price \(r_k\) satisfying \(r_k - \frac{1}{\lambda_k(r_k)} = \frac{1}{c - \tau_k}\). A buyer of type \(\theta\) attends auction \(k\) if and only if \(v_k(\theta) \geq r_k\).

### 4.2.2 Duopolist sellers

When products are less differentiated and sellers’ values are low, sellers compete for buyers in the overlapping market segment by setting reserve prices low. In doing so, sellers leave positive participation rents to the buyer who is indifferent between attending the two auctions. When sellers’ values increase, they raise reserve prices and the overlapping region becomes smaller. Consequently, the participation rents for the indifferent type decrease. In particular, there exists a critical value \(c^*\) such that the indifferent type gets exactly zero rents. However, even at that point, sellers’ effective marginal costs are smaller than marginal revenues. This distinguishes the case from the local monopoly one. The following proposition characterizes the equilibrium outcome and the relevant region of sellers’ values for this outcome.

**Proposition 4.6.** For any given \(\tau\) and \(t < t^*\), there exists a unique \(c^* < c^{**}\) such that for any \(c < c^*\), the following is the equilibrium outcome in the sellers and buyers’ subgame.
1. Sellers set reserve prices satisfying
\[ r_k(c, \tau, t) = v_k(\bar{\theta}) - \frac{1}{N} \left( \psi_k(\bar{\theta}, t) - \frac{c}{1 - \tau_k} \right) \]  
(6)
where \( \psi_1(\theta, t) = MR_1(\theta, t) - t\frac{F(\theta)}{1-F(\theta)} \) \( N-1 \), \( \psi_2(\theta, t) = MR_2(\theta, t) - t\frac{1-F(\theta)}{F(\theta)} \) \( N-1 \), and \( \bar{\theta} \) satisfies
\[ (1 - F(\bar{\theta}))^{N-1} \left( \psi_1(\bar{\theta}, t) - \frac{c}{1 - \tau_1} \right) = F^{N-1}(\bar{\theta}) \left( \psi_2(\bar{\theta}, t) - \frac{c}{1 - \tau_2} \right) \]  
(7)

2. All buyers with \( \theta < \bar{\theta} \) attend auction 1, all buyers with \( \theta > \bar{\theta} \) attend auction 2, and \( \bar{\theta} \) is indifferent between attending the two auctions.

c* is such that c* = (1 - \( \tau_k \))\( \psi_k(\bar{\theta}(c^*), t) \) if \( \psi_k(\bar{\theta}(0), t) > 0 \), and c* = 0 otherwise.

Proof. We proceed in three steps. First, we show the existence and uniqueness of c* with the property stated in the proposition. Second, we show that for c < c*, the condition in Proposition 4.2 holds. This then implies that buyers’ behavior stated in (2) constitutes an equilibrium. Third, we show that sellers’ reserve prices stated in (1) are equilibrium prices.

First, we show that there exists a unique c* \( \in (0, c^{**}) \) such that \( \psi_k(\bar{\theta}, t) - \frac{c^*}{1 - \tau_k} = 0 \), for some \( \bar{\theta} \in (0, 1) \). Since the event c** = \( \bar{c} \) has zero probability, we assume for the following analysis that c** < \( \bar{c} \). Now for any given c < \( \bar{c} \), define functions \( \Delta_k : [0, 1] \rightarrow \mathbb{R} \) by \( \Delta_1(\theta, c) = (1 - F(\theta))^{N-1} \left( \psi_1(\theta, t) - \frac{c}{1 - \tau_1} \right) \), and \( \Delta_2(\theta, c) = F^{N-1}(\theta) \left( \psi_2(\theta, t) - \frac{c}{1 - \tau_2} \right) \).

Our objective is to find a \( \bar{\theta} \in (0, 1) \) such that \( \Delta_1(\bar{\theta}, c) = \Delta_2(\bar{\theta}, c) = 0 \). This is done in the following four steps. Figure 4 illustrates the idea of the proof. Notice at c = 0, we have either \( \bar{\theta}_1(0) \leq \bar{\theta}_2(0) \), in which case we set c* = 0, or \( \bar{\theta}_1(0) > \bar{\theta}_2(0) \), in which case we show the existence and uniqueness of c* in (0, c**) as follows.

1. For any given c < \( \bar{c} \), there exist a unique \( \bar{\theta}_k(c) \) with \( \Delta_k(\bar{\theta}_k(c), c) = 0 \).

Since \( \Delta_1(0, c) = 1 - \frac{c}{1 - \tau_1} > 0 \) and \( \Delta_1(1, c) = -\frac{c}{1 - (1)} < 0 \), there is a unique \( \bar{\theta}_1(c) \in (0, 1) \) such that \( \Delta_1(\bar{\theta}_1(c), c) = 0 \). Similarly, \( \Delta_2(0, c) = -\frac{c}{1 - (0)} < 0 \), \( \Delta_2(1, c) = 1 - \frac{c}{1 - \tau_2} > 0 \), so there is a unique \( \bar{\theta}_2(c) \in (0, 1) \) such that \( \Delta_2(\bar{\theta}_2(c), c) = 0 \).

2. \( \bar{\theta}_1(0) - \bar{\theta}_2(0) > 0 \). This is the case to be considered here.
\[ c = 0 \quad \text{and} \quad c = c^* \]

\[ \theta_1(0), \theta_2(0) \quad \text{and} \quad \theta_1(0), \theta_2(0) \quad \text{respectively} \]

\[ \exists c^* \in (0, c^{**}) \]

\[ \theta_1(c^*) = \theta_2(c^*) \]

\[ \theta_1(0), \theta_2(0) \quad \text{and} \quad \theta_1(0), \theta_2(0) \quad \text{respectively} \]

\[ \text{Figure 4: Proof of the existence of } c^* \]

3. \( \bar{\theta}_1(c^{**}) - \bar{\theta}_2(c^{**}) < 0. \)

Recall that Proposition 4.4 shows \( MR_k(\bar{\theta}(c^{**}), t) = \frac{c^{**}}{1 - \tau_k} \). Since \( \psi_k(\theta, t) < MR_k(\theta, t) \), \( \Delta_k(\theta_k(c^{**})) = 0 \Rightarrow \bar{\theta}_1(c^{**}) < \bar{\theta}(c^{**}), \bar{\theta}(c^{**}) < \bar{\theta}_2(c^{**}) \Rightarrow \bar{\theta}_1(c^{**}) < \bar{\theta}_2(c^{**}). \)

4. \( \frac{\partial}{\partial c}(\bar{\theta}_1(c) - \bar{\theta}_2(c)) < 0. \)

For \( \theta \in (0, 1) \), taking the derivative with respect to \( c \) of the equation \( \psi_k(\bar{\theta}_k(c)) - \frac{c}{1 - \tau_k} = 0 \), we get \( \frac{\partial \bar{\theta}_1(c)}{\partial c} < 0 \) and \( \frac{\partial \bar{\theta}_2(c)}{\partial c} > 0. \)

Combining (1)-(4) implies that there exists a unique \( c^* \in (0, c^{**}) \) such that \( \bar{\theta}_1(c^*) = \bar{\theta}_2(c^*) = (\bar{\theta}(c^*)) \). This shows that \( \Delta_k(\bar{\theta}(c^*), c^*) = 0 \), for \( k = 1, 2 \). Since \( \bar{\theta}(c^*) \in (0, 1) \), this implies that \( \psi_k(\bar{\theta}(c^*), t) - \frac{c^*}{1 - \tau_k} = 0. \)

Second, we show that for \( c < c^* \), reserve prices given by (6)-(7) imply \( \bar{\theta}_2(c) < \bar{\theta}_1(c) \). It then follows directly from Proposition 4.2 that buyers’ behavior stated in (2) constitutes an equilibrium. To see \( \bar{\theta}_2(c) < \bar{\theta}_1(c) \) for \( c < c^* \), simply notice that \( \bar{\theta}_1(c) - \bar{\theta}_2(c) \) is decreasing in \( c \), and \( \bar{\theta}_1(c^*) - \bar{\theta}_2(c^*) = 0. \)

Third, we show that for \( c < c^* \), equilibrium reserve prices are given by (6)-(7). Seller \( k \) solves (2) with \( x_k > r_k \). The first order condition is

\[ 0 = \frac{\partial \Pi_k}{\partial r_k} = N \Phi_k^{N-1}(x_k) \phi_k(x_k) \frac{\partial x_k}{\partial r_k} \left( c - (1 - \tau_k) \left( x_k - \frac{\Omega}{\lambda_k(x_k)} - N(x_k - r_k) \right) \right) \]

where \( \Omega = 1 + \left( \frac{1 - \Phi_k(x_k)}{\Phi_k(x_k)} \right)^{N-1} \). The optimal reserve prices are given by

\[ r_k = x_k - \frac{1}{N} \left( x_k - \frac{\Omega}{\lambda_k(x_k)} - \frac{c}{1 - \tau_k} \right) \quad \text{(8)} \]
Rearranging, we get (6). The indifferent type \( \bar{\theta} \) is characterized by

\[
(1 - F(\bar{\theta}))^{N-1}(v_1(\bar{\theta}) - r_1(c, \tau, t)) = F^{N-1}(\bar{\theta})(v_2(\bar{\theta}) - r_2(c, \tau, t))
\]

Using (6) to replace \( v_k(\bar{\theta}) - r_k(c, \tau, t) \) with \( \frac{1}{N} \left( \psi_k(\bar{\theta}, t) - \frac{c}{1 - r_k} \right) \) yields (7).

The second order condition for a maximum is satisfied, as shown below.

\[
\frac{\partial^2 \Pi_k}{\partial r_k^2} = N \Phi_k^{N-1}(x_k) \phi_k(x_k)(1 - \tau_k) \left( \frac{\partial x_k}{\partial r_k} \right)^2 \left( \Delta \left( \frac{1}{\lambda_k(x_k)} \right) + \frac{1}{\lambda_k(x_k)} \frac{\partial \Delta}{\partial x_k} \right)
\]

\[
+ N \Phi_k^{N-1}(x_k) \phi_k(x_k)(1 - \tau_k) \frac{\partial x_k}{\partial r_k} \left( (N - 1) \frac{\partial x_k}{\partial r_k} - N \right)
\]

The first term on the RHS is negative since \( \frac{\partial}{\partial x_k} \frac{1}{\phi_k(x_k)} < 0 \) and \( \frac{\partial \Delta}{\partial x_k} < 0 \). The second term is also negative since \( 0 < \frac{\partial x_k}{\partial r_k} < 1 \). To see this, recall that for \( r_1, r_2 \) such that markets are overlapping, the indifferent type satisfies (1):

\[
\Phi_1^{N-1}(v_1(\bar{\theta}))(v_1(\bar{\theta}) - r_1) = \Phi_2^{N-1}(v_2(\bar{\theta}))(v_2(\bar{\theta}) - r_2)
\]

where \( v_k(\bar{\theta}) = x_k \). We have \( \frac{\partial \theta}{\partial r_1} < 0, \frac{\partial \theta}{\partial r_2} > 0 \). So \( \frac{\partial x_1}{\partial r_1} = -t \frac{\partial \theta}{\partial r_1} > 0, \frac{\partial x_2}{\partial r_2} = t \frac{\partial \theta}{\partial r_2} > 0 \), and

\[
\frac{\partial x_1}{\partial r_1} = 1 + \frac{\partial \theta}{\partial r_1} \left( \frac{\partial}{\partial \theta} \left( \frac{F(\theta)}{1 - F(\theta)} \right)^N \right)^{-1} (x_2 - r_2) + t \left( \frac{F(\bar{\theta})}{1 - F(\bar{\theta})} \right)^N < 1
\]

\[
\frac{\partial x_2}{\partial r_2} = 1 + \frac{\partial \theta}{\partial r_2} \left( \frac{\partial}{\partial \theta} \left( \frac{1 - F(\theta)}{F(\theta)} \right)^N \right)^{-1} (x_1 - r_1) - t \left( \frac{1 - F(\bar{\theta})}{F(\bar{\theta})} \right)^N < 1
\]

Finally, we check that given seller 2’s equilibrium reserve price, seller 1 has no incentive to deviate. Suppose that given \( r_2 \), seller 1 increases her reserve price to \( v_1(\theta) \) such that markets are separated. Then seller 1 loses buyers in \( [\theta, \bar{\theta}] \), and the marginal revenue that she can get from each of those buyers is higher than her effective marginal cost. Therefore, it is not optimal for seller 1 to deviate.

\[
4.2.3 \quad \text{Constrained duopolist sellers}
\]

We are left with the case where sellers’ values are in the intermediate range, that is, \( c \in (c^*, c^{**}) \). In this case, sellers’ values are neither high enough for them to set monopoly reserve prices, nor sufficiently low for them to overlap each other’s market. Indeed, if each seller simply responds to her increasing values by raising her reserve price, there would be a range
of buyers whose participation rents are negative and hence would not attend any auction. Therefore, sellers are better off setting reserve prices at one of these buyers’ valuations for their items respectively, so long as their marginal revenues from that buyer is no less than their effective marginal costs. There are typically multiple equilibria in this case. We focus on the equilibrium that maximizes sellers’ joint expected profits without further mentioning this selection criteria for the rest of the paper.

**Proposition 4.7.** For any given $\tau$ and $t < t^*$, for any $c \in (c^*, c^{**})$, the following characterizes all equilibrium outcomes in the sellers and buyers’ subgame.

1. Sellers set reserve prices satisfying
   \[ r_k(c, \tau, t) = v_k(\hat{\theta}) \]  
   for any $\hat{\theta} \in \Theta = [\hat{\theta}_1, \hat{\theta}_2] \cap [\hat{\theta}_2, \hat{\theta}_1]$, where $\psi_k(\hat{\theta}_k) = \frac{c}{1-\tau_k}$, $MR_k(\hat{\theta}_k) = \frac{c}{1-\tau_k}$.

2. All buyers with $\theta < \hat{\theta}$ attend auction 1, all buyers with $\theta > \hat{\theta}$ attend auction 2, and $\hat{\theta}$ is indifferent between attending the two auctions.

In particular, there is an equilibrium where sellers maximize their joint expected profits. The indifferent type in that equilibrium is given by

\[ (1 - F(\hat{\theta}))^N - (1 - \tau_1)MR_1(\hat{\theta}, t) - c = F^N - (1 - \tau_2)MR_2(\hat{\theta}, t) - c \]  

**Proof.** In any of the equilibria specified in this proposition, sellers set reserve prices such that the indifferent type $\hat{\theta}$ gets zero rents. All buyers with $\theta < \hat{\theta}$ expect non-negative profits from attending auction 1, and all $\theta > \hat{\theta}$ expect the same from auction 2. This shows that buyers’ behavior stated in (2) constitutes an equilibrium.

We now show that sellers’ strategies form a Nash equilibrium. First, note that in any equilibrium, sellers choose the indifferent type $\hat{\theta}$ from $\Theta$. To see why this must be true, notice for $c \in (c^*, c^{**})$,

(i) $MR_k(\hat{\theta}_k) = \frac{c}{1-\tau_k} \Rightarrow \hat{\theta}_1 > \hat{\theta}_2$. Both sellers are willing to attract buyers from $[\hat{\theta}_2, \hat{\theta}_1]$ since each seller’s marginal revenue from a buyer in that region is greater than her effective marginal cost. If $\theta \notin [\hat{\theta}_2, \hat{\theta}_1]$, one seller’s marginal revenue from $\theta$ is less than effective
marginal cost, hence it is not optimal for that seller to set reserve price at \(\theta\)’s valuation for her item.

(ii) \(\psi_k(\hat{\theta}_k) = \frac{c}{1-r_k} \Rightarrow \hat{\theta}_1 < \hat{\theta}_2\). Seller 1 will not set \(r_1 = v_1(\theta), \theta < \hat{\theta}_1\), since she can get all buyers from \([\theta, \hat{\theta}_1]\) if she were to lower reserve price to \(v_1(\hat{\theta}_1)\). Similarly, seller 2 will not set \(r_2 = v_2(\theta), \theta > \hat{\theta}_2\).

Combining (i) and (ii) shows that if equilibrium reserve prices satisfy (9), then \(\hat{\theta} \in \Theta\).

Next, suppose that seller 2 sets \(r_2 = v_2(\hat{\theta})\), for some \(\hat{\theta} \in \Theta\). We show that seller 1’s best response is to set \(r_1 = v_1(\hat{\theta})\). Seller 1 setting \(r_1 = v_1(\theta_1) > v_1(\hat{\theta})\) is also not a best response to \(r_2\). Suppose the contrary is true. Then, Proposition 4.2 implies that there exists a type \(\bar{\theta} \in (\hat{\theta}, \theta_1)\) who is indifferent between the two auctions, and gets positive rents. So \(r_1 = v_1(\hat{\theta}) - \frac{1}{N} (\psi_1(\hat{\theta}, t) - \frac{c}{1-r_1})\), subject to the constraint \(\psi_1(\hat{\theta}, t) - \frac{c}{1-r_1} > 0\). Since \(r_2 = v_2(\hat{\theta})\) is fixed, seller 1 should choose \(r_1\) such that \(\bar{\theta}\)’s rents \(F^{N-1}(\bar{\theta})(v_2(\bar{\theta}) - v_2(\hat{\theta}))\) is minimized, in other words, \(\hat{\theta} \rightarrow \hat{\theta}\). But since we assume \(\theta_1 > \hat{\theta}\), this minimum cannot be achieved, which is a contradiction to \(r_1\) being a best response. The above argument implies that \(r_1 = v_1(\hat{\theta})\) and \(r_2 = v_2(\hat{\theta})\) form a Nash equilibrium.

Finally, we consider the equilibrium where sellers choose the indifferent type to maximize their joint expected profits. The first order condition to maximize \(\Pi_{1,2} = \Pi_1 + \Pi_2\) is

\[
0 = \frac{\partial \Pi_{1,2}}{\partial \theta} = \sum_{k=1,2} (-1)^k tN\Phi^{-1}_k(r_k)(1 - \Phi_k(r_k))(1 - \tau_k)(1 - r_k\lambda_k(r_k)) + c\lambda_k(r_k))
\]

\[
= NF(\bar{\theta})((1 - F(\bar{\theta}))^{N-1}((1 - \tau_1)MR_1(\bar{\theta}, t) - c) - F(\bar{\theta})^{N-1}((1 - \tau_2)MR_2(\bar{\theta}, t) - c))
\]

Rearranging, we have (10). The second order condition for a maximum is satisfied since \(\frac{\partial^2 \Pi_{1,2}}{\partial \theta^2} < 0\) due to Assumption 1.

Let us summarize the above results. For any given \(\tau\) and \(t < t^*\), there are two critical values \(c^*\) and \(c^{**}\) which divide \([0, \bar{c}]\) into three regions, each giving rise to a different type of equilibrium, which closely parallels the Hotelling-Bertrand model.

**Region 1.** \(c^{**} \leq c \leq \bar{c}\): Local monopolist sellers.
When $c^{**} < c \leq \bar{c}$, sellers are local monopolists in their respective markets. They set reserve prices $r_k = v_k(\tilde{\theta}_k)$, where $\tilde{\theta}_k$ satisfies $MR_k(\tilde{\theta}_k) = \frac{c}{1-\tau_k}$. A buyer of type $\theta$ attends auction $k$ if and only if $v_k(\theta) \geq r_k$. This is illustrated in Figure 5, where the red and blue lines refer to buyers who attend auction 1 and 2 respectively.

When $c = c^{**}$, reserve prices are such that $r_k = v_k(\tilde{\theta})$, where $\tilde{\theta}$ is the type indifferent between the two auctions, and satisfies $MR_k(\tilde{\theta}) = \frac{c^{**}}{1-\tau_k}$. Sellers are still local monopolists, and the market $[0, 1]$ is fully covered. This is illustrated in Figure 6.

![Figure 5: $c^{**} < c \leq \bar{c}$](image1)

![Figure 6: $c = c^{**}$](image2)

**Region 2.** $0 \leq c \leq c^*$: Duopolist sellers.

When $0 \leq c < c^*$, sellers have low values and compete fiercely. Reserve prices are set low such that the participating rents are positive for the type $\bar{\theta}$ who is indifferent between attending the two auctions. Specifically, reserve prices are $r_k = v_k(\bar{\theta}_k)$, where $\bar{\theta}_k$ satisfies $\psi_k(\bar{\theta}_k) = \frac{c}{1-\tau_k}$. This is illustrated in Figure 7.

When $c = c^*$, reserve prices are higher, given by $r_k = v_k(\bar{\theta})$, where $\bar{\theta}$ satisfies $\psi_k(\bar{\theta}) = \frac{c^*}{1-\tau_k}$. That is, the indifferent type $\bar{\theta}$ gets zero rents. This is illustrated in Figure 8. Note that although the market is fully covered, $MR_k(\bar{\theta}) > \frac{c^*}{1-\tau_k}$, so $r_k$ are not monopoly reserve prices.

![Figure 7: $0 \leq c < c^*$](image3)

**Region 3.** $c^* < c < c^{**}$: Constrained duopolist sellers.
When seller’s values are in this intermediate range, there are multiple equilibria. In each equilibrium, sellers set reserve prices $r_k = v_k(\hat{\theta})$, where $\hat{\theta}$ is the type indifferent between the two auctions, and satisfies $\psi_k(\hat{\theta}) \leq e^\frac{\theta}{1-\tau_k} \leq MR_k(\hat{\theta})$. This is illustrated in Figure 9.

\begin{equation} \label{equilibrium} \end{equation}

4.3 The full game

Auctioneer $j$ sets a percentage fee $\tau_j \in [0, 1]$ to maximize his expected profit anticipating which seller to host and the seller’s equilibrium reserve price. Different rationing rules in general lead to different anticipations, which we will discuss soon after specifying auctioneers’ problem. For a given rationing rule, Let $p_{jk}(\tau)$ be the probability that auctioneer $j$ hosts seller $k$ when the percentage fees are $\tau$. Auctioneer $j$ chooses $\tau_j \in [0, 1]$ to maximize

$$\Pi_j = \sum_{k=1,2} p_{jk}(\tau) \int_0^{\tilde{c}_j} \tau_j NM_k(\Phi_k, r_k, r_{-k})dH(c)$$

where $\tilde{c}_j = 1 - \tau_j$ is the highest value for sellers to be willing to participate in the auction subject to percentage fee $\tau_j$.

In the next two subsections, we derive equilibrium percentage fees for capacity constrained auctioneers. Depending on the rationing rule, auctioneers may have incentives to compete for the good seller. That is, assume one seller generates more expected revenue for a given percentage fee. Such incentives do not exist under non-discriminatory rationing rule since an auctioneer is expected to host each seller with equal probability regardless of the other auctioneer’s fee. However, when discriminatory rationing rule is used, auctioneers always
have incentives to undercut each other so long as the fees are above certain threshold. If both auctioneers charge the same fee, then each gets the good seller with probability one-half. Now if one auctioneer slightly lowers his fee, he attracts both sellers and gets the good seller for sure, so his expected profit is strictly higher at this slightly lower fee. Hence both auctioneers have incentives to undercut each other. Before analyzing this case with discriminatory rationing, we first analyze cases where sellers are symmetric in terms of expected revenue they generate, or when the rationing rule is non-discriminatory.

4.3.1 Non-discriminatory rationing

We consider two cases. First, the distribution function $F$ is symmetric. This implies that both sellers generate the same expected revenue if subject to the same percentage fee. We assume that in this case, an auctioneer chooses each seller with equal probability if both attend him. Second, the rationing rule is non-discriminatory, that is, if both sellers go to the same auctioneer, then each is hosted with probability one-half. In both cases, the incentives for auctioneers to undercut each other to get the good seller for sure do not exist. We refer to these situations as *Bertrand-Edgeworth competition with non-discriminatory rationing*.

We denote by $E_G(\tau_j)$ and $E_B(\tau_j)$ the expected revenues from the good seller and the bad seller respectively when the percentage fee is $\tau_j$.

**Theorem 4.8.** For any given $t$, both auctioneers set the same equilibrium percentage fee $\frac{1}{\gamma+1}$ in Bertrand-Edgeworth competition with non-discriminatory rationing.

*Proof.* We first prove the theorem for $t \geq t^*$. Notice each auctioneer’s expected profit is always $\frac{1}{2} \tau_j(E_G(\tau_j) + E_B(\tau_j))$, regardless of the other auctioneer’s fee. And recall that when $t \geq t^*$, we have the local monopoly outcome in the sellers and buyers’ subgame. That is, both the good and the bad seller are local monopolists. Auctioneer $j$’s expected profit is

$$\Pi_j = \frac{1}{2} \int_0^{\tilde{c}_j} \tau_j N(M_1(\Phi_1, r_1) + M_2(\Phi_2, r_2)) \gamma c^{\gamma-1} dc$$

where $r_k$ satisfy

$$r_1 = \frac{c}{1-\tau_j} + t \frac{F(\hat{\theta}_1)}{f(\hat{\theta}_1)} \text{ and } r_2 = \frac{c}{1-\tau_j} + t \frac{1-F(\hat{\theta}_2)}{f(\hat{\theta}_2)},$$

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with \( \tilde{\theta}_1 = \frac{1 - \tau}{t} \), \( \tilde{\theta}_2 = \frac{r - t + t_k}{t} \). \( M_k \) is given by (3) with \( x_k = r_k \). Let \( \mu_j = \frac{c}{1 - \tau_j} \). Using integration by substitution, we obtain

\[
\Pi_j = \frac{1}{2} \tau_j (1 - \tau_j)^\gamma \gamma N \int_0^1 (M_1(\Phi_1, r_1) + M_2(\Phi_2, r_2)) \mu_j^{\gamma-1} d\mu_j \tag{12}
\]

We observe that \( \tilde{\theta}_k \) depends only on \( \mu_j \), so reserve prices \( r_k \) and expected payments \( M_k \) depend only on \( \mu_j \). Therefore, the integral in (12) is independent of \( \tau_j \). Consequently, the first order condition to maximizing (12) can be reduced to \( \frac{\partial}{\partial \tau_j} \tau_j (1 - \tau_j)^\gamma = 0 \), which gives the optimal percentage fee \( \tau_j = \frac{1}{\gamma+1} \). The second order condition for a maximum \( \frac{\partial^2 \Pi_j}{\partial \tau_j^2} < 0 \) is clearly satisfied. This shows that it is a dominant strategy for each auctioneer to set \( \tau_j = \frac{1}{\gamma+1} \) for any \( t \geq t^* \).

Next, assume \( t < t^* \). Propositions 4.4 - 4.7 show that equilibrium outcomes in the sellers and buyers’ subgame depend on \( c \). So we first calculate auctioneer \( j \)’s expected profit in the three regions of \( c \).

**Region 1:** \( c^{**} \leq c \leq 1 - \tau_j \).

Auctioneer \( j \)’s expected profit is

\[
\Pi_j^{I} = \frac{1}{2} \int_{\mu_j^{**}}^{\tilde{\theta}_j} \tau_j N(M_1(\Phi_1, r_k, r_{-k}) + M_2(\Phi_2, r_k, r_{-k})) dH(c)
\]

From Proposition 4.4, we see that \( \mu_j^{**} \) does not depend on \( \tau_j \) since \( \mu_j^{**} = M R_k(\tilde{\theta}(\mu_j^{**}, \mu_{-j}^{**})) \).

Following a similar analysis as in Theorem 4.8, we can write \( \Pi_j^{I} = \tau_j (1 - \tau_j)^\gamma R_1 \), where

\[
R_1 = \frac{1}{2} \gamma N \int_{\mu_j^{**}}^{\tilde{\theta}_j} (M_1(\mu_j) + M_2(\mu_j)) \mu_j^{\gamma-1} d\mu_j
\]

is some positive constant independent of \( \tau_j \).

**Region 2:** \( 0 \leq c \leq c^* \).

Auctioneer \( j \)’s expected profit in this region is

\[
\Pi_j^{II} = \frac{1}{2} \int_0^{c^*} \tau_j N(M_1(\Phi_1(x_1), r_k, r_{-k}) + M_2(\Phi_2(x_2), r_k, r_{-k})) dH(c)
\]

where \( r_k \) is given by (6). Using integration by substitution, we obtain

\[
\Pi_j^{II} = \frac{1}{2} \tau_j (1 - \tau_j)^\gamma \gamma N \int_0^{\mu_j} (M_1(\Phi_1(x_1), r_k, r_{-k}) + M_2(\Phi_2(x_2), r_k, r_{-k})) \mu_j^{\gamma-1} d\mu_j \tag{7}
\]

(7) shows that the indifferent type \( \tilde{\theta} \) is a function of \( \mu_j \) and \( \mu_{-j} \) only. Since \( \mu_j^{*} = \psi_k(\tilde{\theta}(\mu_j^{*}, \mu_{-j}^{*}), t) \), \( \mu_j^{*} \) must be independent of \( \tau_j \). \( M_k \) depends on \( \mu_j \) and \( \mu_{-j} \) only since
\[ r_k, x_k \] are both functions of \( \theta \). Furthermore, \( \mu_j (\mu_{-j}) \) is independent of \( \tau_{-j} (\tau_j) \). Therefore, we can write \( \Pi_{j}^{II} = \tau_j (1 - \tau_j)^\gamma R_2 \), where \( R_2 = \frac{1}{2} \gamma N \int_{\mu_j^*}^{\mu_j} (M_1(\mu_j, \mu_{-j}) + M_2(\mu_j, \mu_{-j})) \mu_j^{-1} d\mu_j \) is some positive constant independent of \( \tau_j \).

**Region 3:** \( c^* < c < c^{**} \).

Auctioneer \( j \)'s expected profit is

\[
\Pi_{j}^{III} = \frac{1}{2} \int_{c^*}^{c^{**}} \tau_j N(M_1(\Phi_1(r_1), r_k, r_{-k}) + M_2(\Phi_2(x_2), r_k, r_{-k})) dH(c)
\]

\[
= \frac{1}{2} \tau_j (1 - \tau_j)^\gamma N \int_{\mu_j^*}^{\mu_j} (M_1(\Phi_1(r_1), r_k, r_{-k}) + M_2(\Phi_2(x_2), r_k, r_{-k})) \mu_j^{-1} d\mu_j
\]

(10) shows that \( \hat{\theta} \) depends only on \( \mu_j \) and \( \mu_{-j} \) since the equation can be rewritten as

\[
(1 - F(\hat{\theta}))^{N-1} \frac{c}{1 - \tau_2} \left( MR_1(\hat{\theta}, t) - \frac{c}{1 - \tau_1} \right) = F^{N-1}(\hat{\theta}) \frac{c}{1 - \tau_1} \left( MR_2(\hat{\theta}, t) - \frac{c}{1 - \tau_2} \right)
\]

(9) shows that \( r_k \) is a function of \( \hat{\theta} \). So \( K_j \) depends on \( \mu_j \) and \( \mu_{-j} \). Therefore, we can write \( \Pi_{j}^{III} = \frac{1}{2} \tau_j (1 - \tau_j)^\gamma R_3 \), where \( R_3 = \frac{1}{2} \gamma N \int_{\mu_j^*}^{\mu_j} M_K(\mu_j, \mu_{-j}) \mu_j^{-1} d\mu_j \) is some positive constant independent of \( \tau_j \).

The first order condition to maximize \( \Pi_j \) can be now reduced to

\[
0 = \frac{\partial}{\partial \tau_j} \tau_j (1 - \tau_j)^\gamma, \text{ which gives the equilibrium percentage fee } \tau_j = \frac{1}{\gamma + 1} \text{ for } t < t^*.
\]

### 4.3.2 Discriminatory rationing

We consider the situation where \( F \) needs not be symmetric, and the rationing rule allows auctioneers to choose seller freely. As explained above, auctioneers will have incentives to undercut each other in order to get the good seller for sure. We call this *Bertrand-Edgeworth competition with discriminatory rationing*. Theorem 4.9 shows that both auctioneers play mixed strategies in equilibrium.

From the previous subsection, we see that expected revenue from an auction under percentage fee \( \tau_j \) is \( (1 - \tau_j)^\gamma \) times some constant, which typically depends on \( F \), the identity of the seller, and the type of equilibrium as a function of \( t \) and \( c \). We label this constant \( R^G \) and \( R^B \) for the good and the bad seller respectively. By assumption, \( R^G > R^B \).
Theorem 4.9. In Bertrand-Edgeworth competition with discriminatory rationing, both auctioneers play only mixed strategies $G_j(\tau_j)$ in equilibrium, where $G_j(\tau_j)$ is an increasing continuous distribution function on the interval $[\tau, \bar{\tau}]$, with $\bar{\tau} = \frac{1}{\gamma+1}$, and $\bar{\tau}$ satisfying $\bar{\tau}(1 - \bar{\tau})^\gamma = \frac{1}{\gamma+1}(1 - \frac{1}{\gamma+1})^\gamma \frac{RB}{RB}$. 

Proof. We prove the theorem in four steps.

1. We first show that there is no pure strategy equilibrium. To see this, note that the upper bound of $\tau_j$ is $\frac{1}{\gamma+1}$ since it maximizes $j$’s expected profit in the worst case scenario where he is undercut for sure and gets the bad seller with probability one. Now suppose $\tau_j = \frac{1}{\gamma+1}$. Then auctioneer $-j$’s best response is to set $\tau_{-j} = \tau_j - \epsilon$, and but then $j$ is better off setting $\tau_j = \tau_{-j} - \epsilon$. This mutual undercutting continues until $\tau_{-j}$ falls below a threshold $\tau$ which satisfies $\tau E_G(\tau) = \frac{1}{\gamma+1} E_B(\frac{1}{\gamma+1})$. In this case $j$’s best response is to set $\tau_j = \frac{1}{\gamma+1}$. But we have already seen that $-j$’s best response to this is to set $\tau_{-j} = \tau_j - \epsilon$, and undercutting starts again. Therefore, no pure strategy equilibrium exists.

2. Assuming that $j$’s equilibrium mixed strategy $G_j(\tau_j)$ is continuous and increasing on $[\underline{\tau}, \bar{\tau}]$. By symmetry, $G_j = G_{-j}$, so we drop subscripts hereafter. The expected profit for $j$ when setting percentage fee $\tau_j \in [\underline{\tau}, \bar{\tau}]$ is

$E\Pi_j(\tau_j) = \tau_j G(\tau_j) E_B(\tau_j) + \tau_j (1 - G(\tau_j)) E_G(\tau_j)$

$G$ has no atom at $\bar{\tau}$ since an auctioneer will be undercut for sure if he sets $\bar{\tau}$, hence it is not optimal to put positive probability on $\bar{\tau}$. We have $\bar{\tau} = \frac{1}{\gamma+1}$ since $\tau_j = \frac{1}{\gamma+1}$ is the best response to $\tau_{-j} < \tau$, and it strictly dominates $\tau_j > \frac{1}{\gamma+1}$: $j$’s expected equilibrium profit is thus $\frac{1}{\gamma+1} E_B(\frac{1}{\gamma+1})$.

3. Setting $\bar{\tau} E_G(\bar{\tau}) = \bar{\tau} E_B(\bar{\tau})$ yields the lower bound $\underline{\tau}$, which satisfies

$\underline{\tau}(1 - \underline{\tau})^\gamma = \frac{1}{\gamma+1} \left(1 - \frac{1}{\gamma+1}\right)^\gamma \frac{RB}{RB}$

Setting $E\Pi_j(\tau_j) = \frac{1}{\gamma+1} E_B(\bar{\tau})$ yields the distribution function

$G(\tau_j) = \frac{\tau_j E_G(\tau_j) - \frac{1}{\gamma+1} E_B(\bar{\tau})}{\tau_j (E_G(\tau_j) - E_B(\tau_j))} = \frac{\tau_j (1 - \tau_j)^\gamma R^G - \frac{1}{\gamma+1} (1 - \frac{1}{\gamma+1})^\gamma R^B}{\tau_j (1 - \tau_j)^\gamma (R^G - R^B)}$
It is easy to check that $G(\tau) = 0$, $G(\bar{\tau}) = 1$, and $G$ is continuous. To see $G$ is increasing, notice

$$\frac{\partial G(\tau_j)}{\partial \tau_j} = \frac{1}{\gamma + 1} (1 - \frac{1}{\gamma + 1})^\gamma R^B (1 - \tau (1 + \gamma)) \geq 0$$

since $\tau \leq \frac{1}{\gamma + 1}$.

4. Finally, we show that there is no profitable deviation for auctioneer $j$ if the other auctioneer follows the equilibrium mixed strategy. This is true since $j$ is indifferent among all $\tau_j \in [\underline{\tau}, \bar{\tau}]$, and if $j$ sets $\tau > \bar{\tau}$ ($\tau < \underline{\tau}$), then he gets the bad (good) seller for sure, but then his expected profit is strictly higher by setting $\tau_j = \bar{\tau}$ ($\tau_j = \underline{\tau}$).

\[\square\]

### 4.4 Monopoly auctioneer with one seller

This can be considered a special situation of the capacity constraint case with non-discriminatory rationing. The results in Section 4.2.1 show that for $t \geq t^*$, or $t < t^*$ and $c \leq 1 - \tau_k$, seller $k$ is the monopoly in her own market, regardless of the presence of a second seller. Consequently, when there is only one seller, the monopoly auctioneer’s equilibrium percentage fee is $\tau = \frac{1}{\gamma + 1}$, following a similar argument as in Theorem 4.8. The full description of this equilibrium outcome is given in the following theorem (recall the seller is located on 1).

**Theorem 4.10.** For any given $t$, the equilibrium in the case of monopoly auctioneer with one seller is as follows.

1. The auctioneer sets percentage fee $\tau = \frac{1}{\gamma + 1}$.

2. If $c \leq 1 - \tau$, the seller participates and sets reserve price $r(c, \tau, t)$ satisfying $r = \frac{c}{1 - \tau} + t \frac{1 - F(\tilde{\theta})}{f(\tilde{\theta})}$, where $\tilde{\theta} = \frac{\tau - 1 + t}{t}$.

3. All buyers with $\theta \geq \tilde{\theta}$ attend the auction, and all other buyers stay out.

### 4.5 Monopoly auctioneer with two competing sellers

When there is a monopoly auctioneer who has the capacity of running two auctions, all results in Sections 4.1 and 4.2 go through with the interpretation of percentage fees being $\tau_1 = \tau_2$ set by the same auctioneer. The auctioneer gets revenues from both auctions, but his
equilibrium percentage fee remains \( \frac{1}{\gamma+1} \) since his expected profit is separately additive on the two expected revenues, and using an argument similar to that in the proof of Theorem 4.8 shows that the percentage fee that maximizes each part is given by \( \frac{1}{\gamma+1} \).

### 4.6 Collusion vs competition

Our results on equilibrium fees set by monopoly auctioneer and competing auctioneers have important policy implications for regulators such as antitrust authorities. We have shown that Bertrand-Edgeworth with non-discriminatory rationing and monopoly auctioneer with competing sellers yield the same equilibrium percentage fee. In particular, the latter case can be interpreted as two auctioneers collusively setting a monopoly percentage fee charged on both sellers. In other words, collusive and competitive percentage fees are the same. This is drastically different from the product market price setting, where collusive prices are higher than competitive prices. To the best of our knowledge, this result is novel, and it indicates that commission fee setting deserves a separate treatment. Using our result, one can argue that the alleged collusive commissions set by Christie’s and Sotheby’s might well arise from their competitive equilibrium behavior, and antitrust authorities should not simply dismiss such a possibility. This certainly points out the relevance and importance of understanding the behavior of intermediating auctioneers.

### 5 Bertrand Model

When there are two auctioneers with no capacity constraints, both auctioneers set percentage fees equal to zero in equilibrium. This is equivalent to the standard Bertrand outcome in the product market models with price competition. We first analyze sellers’ equilibrium responses to different percentage fees. Proposition 5.1 shows that if one auctioneer sets a lower percentage fee, then he attracts both sellers with probability one.

**Proposition 5.1.** Suppose that \( \tau_1 < \tau_2 \), then in equilibrium both sellers choose auctioneer 1 so long as \( c \leq 1 - \tau_1 \), with the possible exceptions at \( c^* \) and \( c^{**} \).

**Proof.** We refer to \( r_k^*(\tau_j) \) as seller \( k \)’s equilibrium reserve price when both sellers are charged
τ_j, and Π_k(r^*_k(·), τ_j) as seller k’s expected profit when she sets reserve price at r^*_k(·) and is charged τ_j. We analyze sellers’ equilibrium behavior for the three regions of c as follows.

**Region 1: c^{**} < c \leq 1 - \tau_1.**

In equilibrium sellers are local monopolists. Each seller’s expected equilibrium profit depends only on her own reserve price. It can be seen from (2) that each seller’s expected equilibrium profit is decreasing in the percentage fee imposed on her. This is because \( \frac{\partial \Pi_k}{\partial \tau_k} < 0 \), so \( \tau_1 < \tau_2 \Rightarrow \Pi_k(r^*_k(\tau_2), \tau_2) < \Pi_k(r^*_k(\tau_1), \tau_1) \). So each seller’s expected equilibrium profit is higher under \( \tau_1 \), hence both go to auctioneer 1.

**Region 2: c^* < c < c^{**}.**

By the same reasoning as above, the joint expected equilibrium profits \( \Pi_{1,2} \) is also decreasing in \( \tau_j \). So \( \Pi_{1,2}(r^*_1(\tau_1), r^*_2(\tau_1), \tau_1) > \Pi_{1,2}(r^*_1(\tau_2), r^*_2(\tau_2), \tau_2) \). This implies that at least one seller is strictly better off under \( \tau_1 \). But if this seller chooses auctioneer 1, then equilibrium implies that the other seller also attends the same auctioneer, since it is her best response to set reserve price at the same indifferent buyer’s valuation for her item, but then she is strictly better off under a lower percentage fee. So both sellers go to auctioneer 1.

**Region 3: 0 \leq c < c^*.**

Let \( \bar{\theta}(\tau_2) \) be the equilibrium indifferent type under \( \tau_2 \). Suppose both sellers attend auctioneer 1, and seller 2 sets reserve price at the supposed equilibrium level, but seller 1 (possibly sub-optimally) sets reserve price such that \( \bar{\theta}(\tau_2) \) remains the indifferent type. That is

\[
\begin{align*}
r_1(\tau_1) &= v_1(\bar{\theta}(\tau_2)) - \frac{1}{N} \left( \psi_1(\bar{\theta}(\tau_2)) - \frac{c}{1 - \tau_1} \right) \\
r^*_1(\tau_2) &= v_1(\bar{\theta}(\tau_2)) - \frac{1}{N} \left( \psi_1(\bar{\theta}(\tau_2)) - \frac{c}{1 - \tau_2} \right)
\end{align*}
\]

where we use \( r_1(\tau_1) \) (without an * ) to indicate that this may not be seller 1’s equilibrium reserve price under \( \tau_1 \). We show that seller 1’s expected profit is still higher
under $\tau_1$. To see this, note that

$$
\Pi_1(r_1(\tau_1), \tau_1) - \Pi_1(r_1^*(\tau_2), \tau_2)
= (\tau_2 - \tau_1) \left( v_1(\bar{\theta}(\tau_2)) - \frac{1}{N} \psi_1(\bar{\theta}(\tau_2)) \right) N(1 - \Phi_1(x_1)) \Phi_1^{N-1}(x_1) \\
+ (\tau_2 - \tau_1) N(N-1) \int_{x_1}^1 y(1 - \Phi_1(y)) \Phi_1^{N-2}(y) d\Phi_1(y)
$$

To show $\Pi_1(r_1(\tau_1), \tau_1) - \Pi_1(r_1^*(\tau_2), \tau_2) > 0$, it suffices to show $v_1(\bar{\theta}(\tau_2)) - \frac{1}{N} \psi_1(\bar{\theta}(\tau_2)) > 0$. This is true since $v_1(\bar{\theta}(\tau_2)) - \frac{1}{N} \psi_1(\bar{\theta}(\tau_2)) = r_1^*(\tau_2) - \frac{c}{N(1-\tau_2)} > 0$ due to the participation constraint $r_1^*(\tau_2) > \frac{c}{(1-\tau_2)} > 0$. So $\Pi_1(r_1(\tau_1), \tau_1) > \Pi_1(r_1^*(\tau_2), \tau_2)$. In particular, $\Pi_1(r_1^*(\tau_1), \tau_1) > \Pi_1(r_1^*(\tau_2), \tau_2)$, that is, seller 1’s expected equilibrium profit is even higher under $\tau_1$. So seller 1 will choose auctioneer 1 since if she does so, then it is not subgame perfect for seller 2 not to choose the same auctioneer. Therefore in equilibrium both sellers go to auctioneer 1.

Combining the above three arguments, we conclude that in all three cases, both sellers choose the auctioneer with a lower percentage fee. 

Notice at $c^*$, $c^{**}$, there are discontinuities in equilibrium reserve prices and expected equilibrium profits for sellers. It is not clear whether a reduction in percentage fees will attract both sellers for sure. However, these two points form an event of probability zero. So when auctioneers are computing their expected profit gains from a reduction in percentage fees, they do not need to worry about the jumps in sellers’ expected equilibrium profits at these two points.

Theorem 5.2 follows easily from sellers’ equilibrium behavior. It shows that competition between auctioneers to attract both sellers drives equilibrium percentage fees to zero.

**Theorem 5.2.** In Bertrand competition, both auctioneers set percentage fees equal to zero in equilibrium.

**Proof.** Proposition 5.1 implies that if an auctioneer sets lower percentage fee, then he gets both sellers with probability one. Therefore, both auctioneers have incentives undercutting each other. It follows a standard argument that this Bertrand type competition reduces equilibrium percentage fees to zero for both auctioneers. 

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6 Conclusions

In this paper, we have provided a general setup to study percentage fee setting by intermediating auctioneers. When seller’s values $c$ are drawn from a power distribution $H(c) = c^\gamma$, equilibrium percentage fees are independent of the number of buyers and the distribution of buyer’s values, whenever there is a monopoly auctioneer, two competing auctioneers with capacity constraints and non-discriminatory rationing, and two auctioneers without capacity constraints. The equilibrium fees in the above cases are all the same, and depend only on the parameter $\gamma$ of the power distribution function, unless they are zero, which happens when there are two unconstrained auctioneers. In Bertrand-Edgeworth competition with discriminatory rationing, auctioneers play the same mixed strategy in equilibrium, with the upper bound of the support of the mixed strategy equal to the equilibrium fee in the previous cases.

An interesting question for future research is to consider percentage fees set by competing auctioneers who can price discriminate. We have already shown that equilibrium fees are lower in Bertrand-Edgeworth competition with discriminatory rationing but without price discrimination. We also have a preliminary result which shows that equilibrium fees (in pure strategies) are on average lower for the good seller when auctioneers can price discriminate than when they cannot. The reason is that competition between auctioneers for the good seller drives down the equilibrium percentage fees levied on her, whereas the fee on the bad seller remains the same as in the current setup without price discrimination and discriminatory rationing. This result implies that allowing auctioneers to price discriminate actually increases the welfare of the good seller since her expected equilibrium profit is higher in the price discrimination case.\footnote{The welfare effects on buyers of an decrease in percentage fees are not straightforward since lower percentage fee leads to lower equilibrium reserve prices, which might actually increases the competition among buyers. See Ginsburgh, Legros and Salhuguet (2005).}

Another extension of the current paper is to explicitly model auctioneers’ capacity choices. Our result on competing auctioneers without capacity constraints shows that equilibrium percentage fees are zero. This is not exactly what we observe in reality, even in the case of internet auctioneers which is closest to Bertrand competition. In our analysis, we have taken capacity choices as exogenously given. It would be more realistic to model these choices...
as strategic variables of auctioneers, as Kreps and Scheinkman (1983) did for the product market with price competition. Here, we briefly discuss the extended setup. The full game would consist of a capacity choice stage which is followed by the game analyzed in the present paper. In the first stage, each auctioneer $j$ chooses the capacity $q_j \in \{1, 2\}$ that allows him to serve one or two sellers. The two cases of competing auctioneers analyzed in the current paper thus correspond to two subgames following the first stage strategy profiles $(1, 1)$ (Bertrand-Edgeworth) and $(2, 2)$ (Bertrand). So we are left to solve the subgame following $(1, 2)$ or $(2, 1)$, i.e. when only one auctioneer is capacity constrained. Let the cost of capacity be $C(q_j)$. It is clear that when $C(2)$ is sufficiently large, then $q_j = 1$ for $j = 1, 2$ is the unique equilibrium capacity choice. So the interesting case is when $C(2)$ is negligible, e.g. when $C(1) = C(2) = 0$. As in Kreps and Scheinkman (1983), $q_j = 2$ for both $j$ would be an equilibrium, because given $q_{-j} = 2$ and $\tau_{-j} = 0$, $q_j = 2$ is always a best response. However, this would not be a subgame perfect equilibrium outcome if the subgame following $(1, 2)$ or $(2, 1)$ yields positive expected profits for the auctioneer with a capacity of 1. Determining the expected equilibrium profits for the asymmetric subgame remains to be done.
References


