THE JOINT HEDGING
AND LEVERAGE DECISION

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ABSTRACT

The validating roles of hedging and leverage as value-adding corporate strategies arise from their beneficial manipulation of deadweight market impositions such as taxes and financial distress costs. These roles may even be symbiotic in their value-adding effects, but they are antithetic in their effects on company risk. This study’s modelling analysis indicates that hedging and leverage do interact for net benefit to company value. However there is no straightforward generalisation for the relative financial riskiness of an optimal joint hedging and leverage strategy in comparison to an unhedged optimal leverage strategy or an unhedged and unlevered strategy; the observed relative riskiness depends on the price level for production output and the company’s remaining production life.

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PREFACE

Title of thesis: The Joint Hedging and Leverage Decision.

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Study of corporate finance has given sparse consideration to the interaction of leverage and hedging decisions. My thesis aims to provide theoretical basis for how leverage and hedging strategies vary when optimised jointly versus individually, and to determine consequential implications for corporate value and financial risk. Towards this aim, the following paper presents a ‘base-case’ modelling analysis of the joint hedging and leverage decision.
1. **INTRODUCTION**

The seemingly patent roles of corporate hedging and leverage are respectively to provide finance and reduce financial risk. However, in getting beyond a Modigliani and Miller (1958) irrelevance argument, their validating roles as value-adding corporate strategies arise from their beneficial manipulation of deadweight market impositions such as taxes and financial distress costs. Expositions in this regard include trade-off theory with respect to leverage, and the work of Smith and Stulz (1985) with respect to hedging.

Ross (1996) took the further step of considering the interrelation of the hedging and capital structure decisions and proposed that hedging facilitates higher optimal leverage and thereby allows firms to access greater tax shield benefits. While hedging and leverage are potentially symbiotic in their value-adding effects, they are antithetic in their effects on company risk: hedging reduces a company’s financial risk ceteris paribus, but it is indeterminate whether the optimal joint hedging and leverage decision should be associated with increased, decreased, or unchanged financial risk compared to that when the company is optimally unhedged and levered, or unhedged and unlevered. This motivates the following research aims:

1. To provide further verification or otherwise of the value-adding benefit of the joint hedging and leverage decision (compared to the unhedged leverage decision and the unhedged and unlevered decision).

2. To provide evidence as to whether the value-maximising joint hedging and leverage decision (compared to the unhedged leverage decision and the unhedged and unlevered decision) is associated with higher, lower or generally unchanged financial risk. Four measures are used to assess financial risk: the value of equity’s comprehensive limited liability option, equity’s beta with respect to the underlying production output’s price, and conditional and unconditional probability of bankruptcy.

The research method utilises a multi-period model for a company subject to respectively hedgeable and unhedgeable production output price and quantity risk variables, endogenously derived deadweight costs, and the tandem availability of risky leverage and flexible hedging control variables. With due concern for the realism of exogenous parameter values, the model is applied as a theoretical tool to investigate both the value and risk impacts of a joint hedging and leverage decision in the presence of deadweight
impositions in the forms of taxation, agency costs of free cash-flow, and costs of financial distress and bankruptcy. The model’s representation of hedging and leverage control and motivation offers favourable innovation compared with previous modelling approaches concerned with the hedging decision in the presence of leverage (e.g. Leland (1998), Mello and Parsons (2000) and Fehle and Tsyplakov (2005)).

A generally qualitative overview of the model company’s design features and application set-up is presented in the following Section 2 for consideration with reference to the exacting design details provided in Appendix A. Section 3 presents and analyses the results of application of the model, and Section 4 concludes the paper.

2. MODELLING APPROACH

I construct a multi-period model of a company, ostensibly a resource producer, subject to hedgeable price risk and unhedgeable quantity risk for its production output. The company has to contend with income tax, agency costs of free cash-flow, and costs of financial distress and bankruptcy, and can manipulate its exposure to these deadweight costs via hedging and leverage control decisions. The model is detailed in Appendix A.

The intention in the model design is to define the company’s equity and non-equity stakes as complex, interdependent, controlled contingent claims (effectively American-style options) on the underlying production output price and quantity random variables \( p_t \) and \( q_t \) respectively. Thereby an option-pricing approach can be used to value equity and non-equity and to assess the effects of hedging and leverage control decisions.

The model company’s non-equity stakeholders are debt finance providers, hedge contract providers, providers of production labour and equipment, and providers of direct bankruptcy services. All non-equity stakeholders provide valuable service without certainty of full compensation in event of bankruptcy, thereby providing equity with its limited liability option. A prominent and important feature of the model design is that at the beginning of every production period, equity purchases from non-equity a fairly priced comprehensive limited liability option exercisable for the ensuing period. The periodic upfront expense for the limited liability option will vary depending on the financial risk being faced by non-equity each period; this expense effectively grosses together the risk premiums that would be charged by individual non-equity stakeholders. The overall non-
equity stake is then defined to be a combination of the risk-free value of the company’s debt plus the short value of the comprehensive limited liability option (which acts to deduct debt’s risk premium as well as the risk premiums of all other non-equity stakeholders from the risk-free debt valuation); the non-equity value is therefore a broader measure than risky debt value.

**Risk variables**

The model company faces two sources of uncertainty: at time $t$, being the end of a discrete production period of duration $\Delta t$ years, the production output for the period is an uncertain quantity of $q_t$ units, and the price obtainable for the production output is an uncertain amount of $p_t$ dollars per unit. The bivariate price and quantity process is assumed to be lognormal Markovian. Hedging can only be contracted with respect to price uncertainty, hence the price risk is hedgeable and the quantity risk is not.

The company is assumed to have uniform production periods, and periodic expected production quantity is assumed to be independent of previous unexpected production quantity deviations. This allows the company to be specified as blind to the history of the stochastic component of production quantities, which advantageously allows equity and non-equity, conditional on price and control variables, to have generalisable analytic valuation solutions with respect to quantity uncertainty.

While the market price for production output ($p_t$) is defined as a lognormal random variable, for the sake of model implementation it is approximated by a binomial process. Within production periods, $p_t$ evolves by an $n$-step recombining binomial tree. However, at the end of each production period the price-tree is specified as non-recombining so that path-dependent hedging and leverage control behaviour is achievable (i.e. there are $(n+1)^{n}$ price-nodes at the end of the first production period, $(n+1)^{2}$ after period two, and so on up to $(n+1)^{N}$ after period $N$). Correlation between production output price and quantity ($\rho$) is used to represent the preference and ability of the company to adjust expected production quantity concurrently with the trend of the market price for its output.
Control variables

The model company is specified to have $N$ production periods for which control decisions can be made, each of duration $\Delta t$ years. Time $(t)$ is denominated accordingly: $t \in \{0, \Delta t, 2\Delta t, \ldots, N\Delta t\}$. The company has available to it a flexible set of hedging and leverage control variables. The periodic leverage decision allows the issue of risky zero-coupon bonds with different maturities and individually specifiable face-values. Similarly the periodic hedging decision allows individually specifiable hedge quantities for different maturities. The available hedge contracts are short forwards, long put options or a ratio combination of the two (however a negative hedge quantity would imply an ‘anti’-hedge consisting of long forwards and/or short put options).

The computational complexity of optimising for all the possible hedging and leverage choices available to the model company compels some limits on control behaviour. The number of controlled production periods is limited to three ($N = 3$). The number of possible output price outcomes for each period is limited to five (i.e. the price evolves by a four-step binomial tree within each period). Hence there are $5^3 = 125$ non-recombining price paths for the three controlled production periods. Also the periodic leverage decision is simplified by restricting the allowable debt maturities to a single production period.

Reiterating from Appendix A, define $y_{t,t+\Delta t} \geq 0$ to be the total face-value of zero-coupon bonds issued at time $t$ and maturing after a single production period; $x_{t,t+\kappa\Delta t}$ to be the hedge quantity contracted at time $t$ and maturing after $\kappa$ production periods; $0 \leq w_{t,t+\kappa\Delta t} \leq 1$ to be the ratio choice anywhere between an all-put hedge ($w_{t,t+\kappa\Delta t} = 0$) and an all-short forward hedge ($w_{t,t+\kappa\Delta t} = 1$); and $z_{t,t+\kappa\Delta t} > 0$ to be the strike price of the put options. The complete set of control variables is: at $t = 0$, one specification of $\{(w_{0,\Delta t}, w_{0,2\Delta t}, w_{0,3\Delta t}); (x_{0,\Delta t}, x_{0,2\Delta t}, x_{0,3\Delta t}); y_{0,\Delta t}; (z_{0,\Delta t}, z_{0,2\Delta t}, z_{0,3\Delta t})\}$; at $t = \Delta t$, five path-dependent specifications of $\{(w_{\Delta t,2\Delta t}, w_{\Delta t,3\Delta t}); (x_{\Delta t,2\Delta t}, x_{\Delta t,3\Delta t}); y_{\Delta t,2\Delta t}; (z_{\Delta t,2\Delta t}, z_{\Delta t,3\Delta t})\}$; and, at $t = 2\Delta t$, 25 path-dependent specifications of $\{w_{2\Delta t,3\Delta t}; x_{2\Delta t,3\Delta t}; y_{2\Delta t,3\Delta t}; z_{2\Delta t,3\Delta t}\}$.

Noteworthy is that, each period, the maturities of new hedge positions (with positive or negative hedge quantities) may overlap with previously established hedge positions so as to increase or decrease the overall hedge position for any particular maturity.

Although the company does not have an explicit option to abandon, it does have the ability to force bankruptcy (and hence abandonment) via an ‘extreme’ hedging or leverage control
decision which radically increases financial risk and thereby triggers immediate bankruptcy. Such voluntary bankruptcy will not necessarily result in loss for non-equity stakeholders and may be desirable when the output price drops so low as to make ongoing production economically unviable.

2.1. Leverage and hedging imperatives

The modelling approach developed for this study is attractive for its inclusion of a range of features that allow the model company to be controlled in respect of several theoretical imperatives. Modigliani and Miller’s (1958) demonstration of the irrelevance of capital structure under the condition of a ‘perfect market’ dictates that acceptance of capital structure relevance must presume deviation(s) from the perfect market condition. Likewise must be the presumption for acceptance of corporate hedging relevance. To this effect, the model company is established as subject to: deadweight costs (taxation) of income attributable to equity; deadweight costs of financial distress; deadweight costs of bankruptcy; and deadweight costs of free cash-flow.

The assumption of asymmetric corporate taxation, entailing tax deductibility of interest payments to lenders but (some degree of) non-deductibility of income attributable to equity, potentially allows firms to add value by using debt finance to reduce their corporate tax burdens. The actual value benefit to any firm of debt finance as a tax shield depends on the corporate tax rate and system (generally classical versus imputation systems), and the personal tax rates for debt and equity income for a ‘marginal’ investor indifferent between debt and equity income. The ‘effective’ rate of tax shielded by an additional unit of corporate debt derives from the additional post-tax value of attributing a dollar of pre-tax corporate income to the marginal investor as an interest payment for debt finance as opposed to a dividend plus capital gain return for equity finance. Under a classical tax system this marginal tax advantage of debt finance over equity finance is:

\[ A = (1 - \tau_{PD}) - (1 - \tau_{C})(1 - \tau_{PE}) \]

where \( \tau_{C} \) is the marginal expected corporate tax rate, and \( \tau_{PD} \) and \( \tau_{PE} \) are respectively the personal tax rates on debt and equity income for the marginal investor. How equity income is split between dividends and capital gains and the investor’s preferences for realising capital gains add complexity to the determination of \( \tau_{PE} \).
The actual process by which any tax benefit of debt financing is capitalised into the value of a firm has two opposing elements: the cash expected to be saved from corporate tax (i.e. higher expected post-corporate tax, pre-financing cash-flows); and a market cost for debt finance that has been grossed-up to reflect the marginal investor’s personal tax penalty for debt income relative to equity income (assuming \(\tau_{PD} > \tau_{PE}\)).\(^1\) To establish whether there is any tax benefit to be had from debt financing, it is problematic that the marginal investor’s characteristics are not specifically ascertainable. Additionally confounding is that the relevant corporate tax rate is the marginal expected rate after consideration of all in-place and expected tax shields. An individual firm’s marginal expected corporate tax rate will be an idiosyncratic function of the convolutions of tax law, such as carry-forward and carry-back tax shield provisions for earnings losses, applied to past and expected future earnings.

Under the taxation regime established for the model company, so long as the company is not bankrupt, positive earnings before tax is subject to the full corporate tax rate (\(\alpha\)), while negative earnings before tax is subject to a partial corporate tax rate (\(\lambda\alpha\), where \(0 \leq \lambda \leq 1\) represents the claimability of a tax refund in event of an earnings loss). That is, instead of allowing carry-forward or carry-back of earnings losses as tax shields, the model’s taxation regime gives the company a partial but immediate tax benefit (refund) for an earnings loss. The tax shields that are available each production period are simply the period-specific expenses used to calculate earnings. The resulting model set-up is such that the company’s marginal expected corporate tax rate arises and adjusts endogenously through time in respect of the risk variables and in response to control decisions.

The model company’s primary non-debt tax shield is total production costs. The company can be considered to lease (for operation) rather than own the necessary physical assets for production, hence depreciation tax shields are implicitly included in total production costs. Hedging outcomes net of transaction costs are also included in taxable earnings. Otherwise the company’s debt tax shield is incorporated as part of a broader expense calculated each period and symbolised by \((y_{t+\Delta t} - Y_{t})\) at time \(t\), where: \(y_{t+\Delta t}\) is the face-value of newly issued debt (with maturity of a single production period); and \(Y_{t}\), termed the risky measure of new debt proceeds, equals the short value of equity’s comprehensive limited liability

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\(^1\) As a generalisation of tax law, relatively favourable personal tax conditions for capital gains mean that the overall personal tax rate for equity income can be expected to be lower than for debt income.
option \((O_t \leq 0)\) plus the risk-free value of the newly issued debt with personal tax penalty adjustment. Effectively the interest expense for debt is calculated and tax deducted immediately upon issue in combination with the expense for the limited liability option each period.

The effective marginal debt tax shield rate \((A)\) for a firm with high demand for tax shields (i.e. for which the marginal expected corporate tax rate equals the full corporate tax rate, \(\tau_c = \alpha\)) indicates the upper limit for that part of the full corporate tax rate that effectively gets shielded by debt finance relative to equity finance after lenders are compensated for their personal tax penalty. Defining \(A_\alpha = A(\tau_c = \alpha)\), the part of the full corporate tax rate that is converse to \(A_\alpha\) (i.e. \((\alpha - A_\alpha)\)) is that part of the full corporate tax rate shield that gets passed on to lenders as a higher rate of return in compensation for their personal tax penalty. Dictating the model company to be valued on a pre-personal tax on equity basis, the company’s proceeds from issuing debt are adjusted for debt’s personal tax penalty via the risk-free debt valuation component of the risky measure of new debt proceeds:

\[
Y_t = O_t + \frac{Y_{t,t+\Delta} \left[1 - (\alpha - A_\alpha)\right]}{e^{r\Delta} - (\alpha - A_\alpha)}
\]

where \(r\) is termed the risk-free interest rate but is more correctly described as the pre-personal tax risk-free rate of return for equity.

As a result of the model’s overall taxation set-up, no direct assumption is made about whether there will be any net tax benefit from any level of debt finance at any stage of the company’s operations. The net tax benefit to be had from each incremental dollar of debt finance, whether positive, zero or negative, arises endogenously, but is, however, affected by the exogenous choices for the \(\lambda\), \(\alpha\) and \(A_\alpha\) parameters. A higher value for the claimability of a tax refund for a loss \((\lambda)\) increases the marginal expected corporate tax rate and makes debt finance more attractive (as a result of the tax rate for a loss being higher, meaning a higher tax refund in event of a loss and lower potential financial penalty

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\(2\) The risk-free debt valuation component of risky new debt proceeds \((Y_t - O_t)\) is adjusted for debt’s personal tax penalty by solving the risk-free valuation inclusive of a tax penalty charge against the income earned by the lender: \((Y_t - O_t) = \left[y_{t,t+\Delta} - y_{t,t+\Delta} - (Y_t - O_t)\right](\alpha - A_\alpha)\).
from being highly levered); similarly a higher value for the full corporate tax rate ($\alpha$) makes debt finance more attractive (for its tax shield effect); whereas a higher value for the personal tax penalty against debt’s corporate tax shield rate ($\alpha - A_{\alpha}$) increases the cost of debt finance and makes it less attractive.

Due to the tangle of details routinely associated with tax law, an empirically appropriate value for $\alpha$ may be less than perfectly manifest. For example, under the US classical tax system the federal corporate tax rate varies across rising earnings bands to become a flat rate of 35% for earnings above a relatively modest threshold of about $18$ million; additionally more irksome is the raft of different state corporate tax rates that can apply (but which are deductible at the federal level). There is also considerable empirical ambiguity about what are appropriate values for $\lambda$ and $A_{\alpha}$. The intention for $\lambda$ is to reflect, on average, corporate tax law provisions for the carry-forward and carry-back of earnings losses for tax shield purposes. Under the US tax system an earnings loss can be carried backward for up to three years and forward for up to 15 years. Hence a US company that is generally profitable suffers relatively little opportunity cost for excess tax shields in event of an earnings loss, which tends to support a relatively high value for $\lambda$; but in turn this may encourage more aggressive use of tax shields (i.e. leverage), making earnings losses more likely and increasing the expected opportunity cost of excess tax shields, thereby lowering the appropriate value for $\lambda$.

Evidence on historic values for $A$ for the US tax system was obtained by Graham (1999) using a simulation procedure and meticulous application of historic tax laws to estimate marginal expected corporate tax rates for individual firms on the COMPSTAT database for each year from 1980 to 1994. This was done with careful consideration of non-debt tax shields for both a before and after debt financing basis. Each year’s median value for the marginal expected corporate tax rate after non-debt tax shields but before debt financing was consistently at or very close to the top corporate tax rate (i.e. $\tau_{C, before debt finance} \approx \alpha$).

With further careful assumptions about the marginal investor’s personal tax rates, Graham also estimated the yearly marginal tax advantage of debt finance for individual firms before any debt finance ($A(\tau_{C, before debt finance})$) and after actual historic debt finance ($A(\tau_{C, after debt finance})$). The median value of $A(\tau_{C, before debt finance})$ ranged between 0.075 and 0.102 for the years 1982 to 1994, but was considerably lower, though still positive, for 1980 and 1981; this indicates a generally consistent and substantial tax advantage for debt
finance. Furthermore, the median value of \( A(\tau_{C, \text{after debt finance}}) \) ranged between -0.002 and 0.018 for the years 1986 to 1994, and was 0.012 for 1980 and 1981, but was notably higher for 1982 to 1985; this is generally consistent with a trade-off theory explanation being that firms take on debt finance up to the point where the marginal tax benefit is zero or, due to offsetting marginal disbenefits, slightly positive.\(^3\)

Trade-off theory suggests that firms will individually take on debt finance up until the marginal expected benefit equals marginal expected disbenefits. The most prominently espoused disbenefit to firms of higher debt levels is higher likelihood of suffering costs of financial distress and bankruptcy.\(^4\) Andrade and Kaplan (1998) investigated a sample of firms that became financially distressed after undertaking highly leveraged transactions (HLTs, i.e. capital restructuring that greatly increased leverage) during the 1980s. To isolate financial distress from economic distress, Andrade and Kaplan limited their sample to firms that maintained positive operating margins while financially distressed. They estimated the costs of financial distress to be about 10% to 20% of pre-distress firm value. For comparison’s sake, in contrast to the broad costs of financial distress, the direct costs of bankruptcy were estimated by Weiss (1990) to be 3.1% of firm value on average. While these cost estimates clearly represent a considerable ex-post encumbrance, the probability of a typical firm experiencing financial distress or bankruptcy is small, and thus so are the expected costs of financial distress and bankruptcy.

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\(^3\) That the trade-off equilibrium for leverage should occur when the marginal tax benefit of debt is zero or slightly positive is consistent with Miller’s (1977) suspicion that the expected bankruptcy cost disbenefit of debt is a minor concern. Specifically Miller described the capital structure trade-off between tax gains and bankruptcy costs as looking like “the recipe for the fabled horse-and-rabbit stew - one horse and one rabbit”!

\(^4\) Financial distress entails difficulty or inability to satisfy financial obligations in a timely manner. There can be several mechanisms by which financial distress reduces the value of a firm: trade creditors, customers and employees who have a stake in the firm as a going concern and fear bankruptcy may behave more restrictively towards the firm; managerial time and resources will be diverted to the financial predicament; some debt covenants may be broken leading to financial or operational penalties; profitable investment opportunities may have to be foregone since new finance will be more difficult to obtain; and valuable assets may have to be sold at depressed prices. These deadweight costs of financial distress will also generally apply once an official state of external administration or bankruptcy is declared. There will also be additional explicit (direct) legal and administrative costs associated with bankruptcy.
For the model company, occurrence of financial distress or bankruptcy, as distinct from a state of solvency, is signalled by the value of periodic free cash-flow ($F_t$), being the overall net cash-flow from operations, hedging, debt and taxation resulting ultimately from control decisions and the outcomes for $p_t$ and $q_t$. Positive or zero free cash-flow is a state of solvency, in which case the free cash-flow is paid out as a dividend to equity (i.e. it is assumed that the company does not retain any earnings or any capital from the issue of debt). Negative free cash-flow signals financial distress and necessitates new equity finance (i.e. a negative dividend, which can be conceptualised as a rights issue). Bankruptcy (and consequential liquidation) occurs when the free cash-flow shortfall is so large that the required amount of new equity finance cannot be justified by the ongoing equity value ($E_{tv}$). By this method the states of financial distress and bankruptcy are determined endogenously.

In event of negative free cash-flow ($F_t < 0$), for the model company to be able to access future earnings potential it must finance the free cash-flow shortfall ($-F_t$); in such case the trade-off optimum for new debt finance takes into account the costliness of resorting to new equity finance. Note that free cash-flow includes cash-flow from new debt finance but excludes cash-flow from new equity finance. The model is set up to appeal to Myers’ (1984) pecking order theory to the extent that endogenous resorting to new equity finance is defined to signify financial distress, and accordingly a financial distress cost is applied with such occurrence. This pushes the leverage trade-off optimum more in favour of debt finance to cover a free cash-flow shortfall and makes new equity finance more of a last resort. The financial distress cost is applied as a factor ($\gamma \geq 0$) of the free cash-flow shortfall so that equity-holders must invest $-F_t(1 + \gamma)$ to finance a free cash-flow shortfall

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5 The leverage trade-off optimum will also be forward looking and give balance to the risk and cost of having to seek new equity finance in the future. Thus there can be trade-off value in maintaining pecking order borrowing capacity. Titman and Tsypolakov (2004) intuited benefit in having “an option to issue debt in the future”. It may even sometimes be optimal to break pecking order and defer new debt finance for new equity finance, despite the deadweight cost, so as to avoid a more severe dependence on new equity finance in the future. Myers (1984) noted that “financial slack (liquid assets or reserve borrowing power) is valuable, and the firm may rationally issue stock to acquire it”. Fama and French (2002) attributed this as forward-looking, “complex” pecking order behaviour entailing “soft”, one-sided leverage targeting (i.e. the financial slack argument will never advocate an increase in leverage).
of \(-F_t\) so as to access an ongoing equity value worth \(E^*_t\). The financial distress cost \((-\gamma F_t)\) is a penalty to equity that represents, for instance, the direct transaction costs of the new equity issue and the operational difficulties that may arise when stakeholders fear imminent bankruptcy; or, more characteristic of pecking order theory, it can represent the costs of assuaging principal-agent information asymmetry.

The endogenous occurrence of bankruptcy for the model company represents equity exercising its limited liability option to refuse new equity finance and liquidate the company. This will occur when the required amount of new equity finance inclusive of financial distress costs \((-F_t(1+\gamma)\) given \(F_t < 0\)\) is greater than the company’s ongoing equity value (i.e. when \(F_t < -E^*_t/(1+\gamma)\)). In this case the company is liquidated for the current period’s operating profit and net payoff of maturing hedge contracts, plus the net market value of any non-maturing hedge contracts, minus the face-value of outstanding debt factored up by a bankruptcy cost rate \((b)\). A positive liquidation cash-flow is a remainder after the claims of all non-equity stakeholders have been satisfied in full; positive liquidation cash-flow is taxed and then paid as a liquidating dividend to equity. A negative liquidation cash-flow indicates a combined loss suffered by non-equity stakeholders due to their claims being partly or wholly unpaid.

It is assumed that only debt-holders are willing to pay anything for bankruptcy proceedings, hence the model’s bankruptcy cost is applied as a factor \((b)\) of the face-value of outstanding debt. Consequently if there is no outstanding debt, there will be no bankruptcy cost. And because the conditions of financial distress and bankruptcy cannot occur concurrently for a production period, there is no financial distress cost applied with the occurrence of bankruptcy. Thus it may be desirable to set the bankruptcy cost rate to a level that incorporates some degree of financial distress cost associated with bankruptcy. Nevertheless, because the financial distress cost rate lowers the level of free cash-flow shortfall at which bankruptcy occurs, there is an effective financial distress opportunity cost associated with bankruptcy (i.e. the company gets liquidated ‘too early’ at a free cash-flow shortfall of \(E^*_t/(1+\gamma)\) instead of \(E^*_t\)).

Deadweight financial distress and bankruptcy costs cause an asymmetry in the possible financial outcomes for a firm such that bad future financial states of the world will be more significantly value-destroying than good future states will be value-enhancing. Hedging, in essence, reduces the variability of future aggregate cash-flows and thereby reduces the
likelihood of both good and bad future financial states (i.e. reduces financial risk). This reduces the firm’s exposure to financial outcome asymmetry and provides a net increase in value equal to the reduction in expected financial distress and bankruptcy costs. This source of added-value from hedging was proposed by Smith and Stulz (1985). Hedging’s effect on financial risk means that it can also be used to ramp-up leverage’s trade-off equilibrium. Ross (1996) proposed that hedging facilitates higher optimal leverage and thereby allows firms to access greater tax shield benefits. Ross explained that hedging “enables the firm to substitute tax-benefitted risk, in the form of leverage, for non-tax-benefitted risk”. Graham and Rogers (2002) were the first to empirically investigate the hedging and leverage decisions jointly and, in favour of Ross’s proposition, they found that firms do hedge to increase debt capacity and that this behaviour is tax motivated. Additionally their evidence indicated that firms also hedge in response to large expected financial distress costs. Graham and Rogers concluded that “a complete modelling of corporate debt policy should control for the influence of hedging decisions”.

In addition to its role as a tax shield, another avenue by which debt can add value is as a shield against agency costs. This potential benefit arises out of Jensen’s (1986) free cash-flow theory which posits that the cash-flow discipline required to sustain high leverage means that there is less free cash-flow available for self-interested managers to squander. Myers (2001) suggested that, with hindsight, it seems clear that the 1980s spate of leveraged buyouts was a manifestation of Jensen’s free cash-flow problem. In reflection of this, the model company is subject to a deadweight agency cost applied as a factor ($0 \leq a \leq 1$) of positive free cash-flow; hence under condition of solvency (i.e. positive or zero free cash-flow, $F_t \geq 0$), management misappropriates an amount equal to $aF_t$, and the dividend paid out to equity equals $(1-a)F_t$.

In summary, the model company serves as a framework by which several prominent corporate finance theories can interact and counteract. Fundamentally it allows playing-out of the trade-off theory of leverage with deference to pecking order theory in regard to dependence on new equity finance. Very importantly the trade-off set-up is augmented to allow playing-out of the effects of hedging: directly on expected financial distress and bankruptcy costs as per Smith and Stulz (1985); and in feedback to trade-off leverage capacity as per Ross (1996). Additionally the trade-off set-up is subject to a deadweight cost of positive free cash-flow as per Jensen’s (1986) free cash-flow theory.
2.2. Valuation method

Assuming a complete market for the output price and perfectly diversifiable production quantity risk, risk-neutral valuation is used to obtain market values for equity and for non-equity’s combined debt plus short limited liability option position. The valuation process works backwards through the binomial price-tree with the production quantity risk incorporated analytically into the valuations at each price-node.

Further to assuming that the model company will operate (while not bankrupt) for three consecutive controlled production periods, it is assumed that the company has presently completed an uncontrolled (i.e. without hedging or leverage) production period, and will operate (if not previously bankrupted) for a fourth and final production period uncontrolled. The presently (time $t=0$) completed production period provides the company with current income which motivates an immediate tax-reducing capital restructure (because the interest expense for leverage is specified within the model as tax deductible in advance). The fourth production period provides an output ‘bank’ from which production can be brought forward or to which production can be delayed,\textsuperscript{6} consequently it is necessary to specify a total output resource ($R$). The binomial price-tree process applied for the three controlled production periods is assumed to continue for the fourth period.

The valuation process requires predetermined valuations at the terminal price-nodes. To this end it is assumed that the output resource remaining to be produced in the final uncontrolled production period is deterministic on the price-path-specific expected production quantities for the prior three controlled production periods. The binomial price-tree is four-step recombinining within production periods and non-recombinining between production periods. Hence there are 125 path-specific price-nodes at the end of the third controlled production period. An ongoing equity value for each of these 125 price-nodes at end-of-period three (start-of-period four) is calculated according to an assumed all-equity payoff exposure for period four:

\textsuperscript{6} Specification of the correlation between production quantity and output price ($\rho$) determines how the company shifts production through time.
\[ E_{3\Delta r} = \max \left[ 0, e^{-r\Delta t} \left( R - \sum_{i=1}^{3} \hat{E}_{(i-1)\Delta t} \left( q_{i\Delta t} | p_{i\Delta t} \right) \right) \hat{E}_{3\Delta t} \left( (p_{4\Delta t} - c) \left( 1 - \alpha \hat{\lambda} - \alpha(1 - \lambda) I_{p_{4\Delta t} > c} \right) \right) \right] \]

where: \( \hat{E} \) is the risk-neutral expectation operator; the indicator function, \( I_{\text{logical statement}} \), equals one if the logical statement is true and zero otherwise; \( R \) is the total output resource for the three controlled and final uncontrolled production periods; \( r \) is the annual, continuously compounding risk-free interest rate; \( c \) is the total production cost per unit of expected production quantity; \( \alpha \) is the corporate tax rate; and \( \hat{\lambda} \) is the claimability of a tax refund in event of negative earnings.

For model set-up it has been assumed that deviation in production quantity from expectation (conditional on price outcome) for a production period does not impact on future production quantity expectations. For the context of a resource producer with a limited resource, the quantity risk does not represent variability of production capacity or efficiency, but rather represents variability of resource quality (e.g. ore concentration) during a production period, albeit with no implication for expectations of resource quality for future production periods. Hence the output resource remaining for fourth period production is calculated by subtracting from the initial total output resource \( R \) the (price-path conditional) expected rather than actual production quantities for the preceding three controlled production periods \( \left( \sum_{i=1}^{3} \hat{E}_{(i-1)\Delta t} \left( q_{i\Delta t} | p_{i\Delta t} \right) \right) \).

Having calculated the 125 path-specific end-of-period three ongoing equity values \( E_{3\Delta r} \), and given the 25 path-specific sets of control variables applicable to period three \( \{(w_{0,3\Delta r}, w_{3\Delta r,3\Delta r}, w_{2\Delta r,3\Delta r}); (x_{0,3\Delta r}, x_{3\Delta r,3\Delta r}, x_{2\Delta r,3\Delta r}); y_{2\Delta r,3\Delta r}; (z_{0,3\Delta r}, z_{3\Delta r,3\Delta r}, z_{2\Delta r,3\Delta r})\} \), it is possible to calculate 125 corresponding price-conditional expected equity and non-equity valuations for the instant immediately prior to end-of-period three. These expected valuations, \( \hat{E}_{2\Delta t} [E_{3\Delta r} | p_{3\Delta r}] \) and \( \hat{E}_{2\Delta t} [D_{3\Delta r} + O_{3\Delta r} | p_{3\Delta r}] \), are respectively obtained using equations (A.1) and (A.2) in Appendix A, where: \( E_{3\Delta r} \) is the total equity value immediately prior to any dividend payment or new equity finance; \( D_{3\Delta r} = y_{2\Delta r,3\Delta r} \) is the risk-free debt value immediately before settlement of the maturing debt; and \( O_{3\Delta r} \) is the

---

7 There are 25 path-specific price-nodes at end-of-period two, for each of which unique control decisions can be made entering into period three.
short payoff value of the expiring comprehensive limited liability option. The price-
conditional expectations are obtained analytically with respect to production quantity
uncertainty and take into account the condition of the company as variously solvent,
financially distressed or bankrupt across the spectrum of possible quantity production
outcomes. The discounted unconditional expectations then give the 25 path-specific end-
of-period two ongoing equity and non-equity values. These valuations, \( E_{2\Delta^+} \) and
\( D_{2\Delta^+} + O_{2\Delta^+} \), are respectively obtained using equations (A.3) and (A.4) in Appendix A,
where: \( E_{2\Delta^+} \) is the ongoing equity value immediately after any dividend payment or new
equity finance and implicitly conditional on bankruptcy having not occurred; \( D_{2\Delta^+} = y_{2M,3\Delta} e^{-r\Delta} \) is the risk-free value of newly issued debt; and \( O_{2\Delta^+} \) is the short initial
value of the newly contracted comprehensive limited liability option.\(^8\)

Now with the 25 path-specific values for \( E_{2\Delta^+} \) and \( D_{2\Delta^+} + O_{2\Delta^+} \), and given the five path-
specific sets of control variables applicable to period two \( \{(w_{0,2\Delta}, w_{0,3\Delta}, w_{2\Delta,2\Delta}, w_{3\Delta,3\Delta}); \ (x_{0,2\Delta}, x_{0,3\Delta}, x_{2\Delta,3\Delta}, x_{3\Delta,3\Delta}); \ (y_{2\Delta,2\Delta}, (z_{0,2\Delta}, z_{0,3\Delta}, z_{2\Delta,3\Delta}, z_{3\Delta,3\Delta}) \} \), the 25 path-specific values
for \( \hat{E}_{\Delta^+}[E_{2\Delta^+} \mid p_{2\Delta}] \) and \( \hat{E}_{\Delta^+}[D_{2\Delta^+} + O_{2\Delta^+} \mid p_{2\Delta}] \) can be calculated (using again equations
(A.1) and (A.2) in Appendix A), and thence the five path-specific values for \( E_{\Delta^+} \) and
\( D_{\Delta^+} + O_{\Delta^+} \) obtain (using again equations (A.3) and (A.4) in Appendix A). Note that the
maturing hedge control variables \( \{(w_{0,2\Delta}, w_{2\Delta,2\Delta}); \ (x_{0,2\Delta}, x_{2\Delta,3\Delta}); \ (z_{0,2\Delta}, z_{2\Delta,3\Delta}) \} \) are
relevant to the valuation process for period two, as are the non-maturing hedge control
variables \( \{(w_{0,3\Delta}, w_{3\Delta,3\Delta}); \ (x_{0,3\Delta}, x_{3\Delta,3\Delta}); \ (z_{0,3\Delta}, z_{3\Delta,3\Delta}) \} \) via their effect on the liquidation
cash-flow in event of bankruptcy at end-of-period two.

Continuing in the same vein, \( E_{\Delta} \) and \( D_{\Delta} + O_{\Delta} \) obtain. With the further assumption of an
uncontrolled production period having just been completed at time \( t = 0 \) with unresolved
production quantity uncertainty, the values \( \hat{E}_{\Delta}[E_{\Delta} \mid p_0] \) and \( \hat{E}_{\Delta}[D_{\Delta} + O_{\Delta} \mid p_0] \) obtain
with \( D_{\Delta} = 0 \) due to the production period being uncontrolled. The combined value of

---

\(^8\) Recall that each production period’s leverage control decision has been restricted to the use of debt with a
maturity of only a single period, and that, by model design, equity’s comprehensive limited liability option
must be newly contracted each period with a maturity of only a single period.
\( \hat{E}_{-\Delta t}[E_{0-} + O_{0-} \mid p_{0}] \) is then used to represent an all-equity valuation that encompasses the impacts of each price-path-dependent control decision going forward.

### 2.3. Model application

To determine the valuation and financial risk differentiations for the model company between a joint hedging and leverage decision, an unhedged leverage decision, and an unhedged and unlevered decision, three strategies for control behaviour are specified. Each strategy has the common aim of maximising the value of the company (signified by \( \hat{E}_{-\Delta t}[E_{0-} + O_{0-} \mid p_{0}] \)), but with different limitations on the available control variables. The control decision to abandon (voluntarily liquidate) the company is made available to all control strategies by always allowing an extreme hedge choice of short forward 10 million units at the start of each production period (irrespective of whether an individual control strategy specifies availability of hedging control). The three control strategies are:

1. **Unlevered, unhedged** strategy, which requires optimisation of the abandonment decision, with the absence of leverage or hedging.

2. **Levered, unhedged** strategy, which requires joint optimisation of the abandonment decision, and the 31 price-path-specific leverage control variables \( y_{t,i+\Delta t} \), with the absence of hedging.

3. **Levered and hedged** strategy, which requires joint optimisation of the abandonment decision, and all 145 price-path-specific leverage and hedging control variables: 31 leverage control variables \( y_{t,i+\Delta t} \); and 38 each of hedge quantity \( x_{t,i+\Delta t} \), short forward versus put option ratio \( \kappa_{t,i+\Delta t} \), and put option strike price \( z_{t,i+\Delta t} \) control variables.

**Measures of leverage, hedging and financial risk**

The three control strategies are assessed for differences in optimised leverage and hedging levels, financial risk and valuation for the model company. Appendix B details the formulations of the leverage, hedging and financial risk measures.
The leverage measure \( \ell_{t+} \) is the risk-free valuation of newly issued bonds \( D_{t+} = y_{t+}\Delta e^{-r\Delta t} \) divided by ongoing equity value \( E_{t+} \). Recall that although the model accommodates risky debt, debt’s risk premium is indistinctly incorporated into the comprehensive limited liability option purchased by equity from all non-equity stakeholders. For this reason \( \ell_{t+} \) is calculated using risk-free debt valuation.

Naive measures of the extent of hedging consider only the contracted price and quantity of individual hedge positions in a firm’s overall hedge portfolio. An alternative approach is to calculate the delta of the hedge portfolio (i.e. the sensitivity of the hedge portfolio value with respect to the underlying asset price) which intrinsically takes into account any non-linearity of the individual hedge positions. The extent of hedging is then indicated by the ratio of the hedge portfolio delta to the quantity of underlying asset to which the firm has financial exposure. This measure is here termed the hedge-delta ratio \( h_{t+} \) and is effectively equivalent to the delta-percentage measure defined by Tufano (1996). The model company’s hedge portfolio delta is calculated as a discrete measure in accordance with the assumed binomial price process; the formulation for the discrete binomial process delta of individual hedge positions \( \Delta x_{t+} = \Delta x_{t+} X_{t+} / \Delta p_{t+} \) is given in Appendix B.

Summing the individual hedge position deltas to give the hedge portfolio delta, the hedge-delta ratios \( h_{t+} \) for each of the company’s three controlled production periods are thus specified to be:

\[
\begin{align*}
 h_{0t+} &= \frac{-\left( \frac{\Delta}{\Delta p_0} x_{0,\Delta} X_{0,0,\Delta} + \frac{\Delta}{\Delta p_0} x_{0,2\Delta} X_{0,0,2\Delta} + \frac{\Delta}{\Delta p_0} x_{0,3\Delta} X_{0,0,3\Delta} \right)}{R}, \\
 h_{\Delta t+} &= \frac{-\left( \frac{\Delta}{\Delta p_{\Delta}} x_{0,2\Delta} X_{\Delta,0,2\Delta} + \frac{\Delta}{\Delta p_{\Delta}} x_{0,3\Delta} X_{\Delta,0,3\Delta} + \frac{\Delta}{\Delta p_{\Delta}} x_{2\Delta,0,2\Delta} X_{\Delta,0,2\Delta} + \frac{\Delta}{\Delta p_{\Delta}} x_{2\Delta,3\Delta} X_{\Delta,0,3\Delta} \right)}{R - \hat{E}_{\Delta} [q_{\Delta} \mid p_{\Delta}]}, \\
 h_{2\Delta t+} &= \frac{-\left( \frac{\Delta}{\Delta p_{2\Delta}} x_{0,3\Delta} X_{2\Delta,0,3\Delta} + \frac{\Delta}{\Delta p_{2\Delta}} x_{2\Delta,3\Delta} X_{2\Delta,0,3\Delta} + \frac{\Delta}{\Delta p_{2\Delta}} x_{2\Delta,3\Delta} X_{2\Delta,2\Delta,3\Delta} \right)}{R - \hat{E}_{0} [q_{\Delta} \mid p_{\Delta}] - \hat{E}_{\Delta} [q_{2\Delta} \mid p_{2\Delta}]}.
\end{align*}
\]

Four measures of financial risk are considered, the first being a limited liability risk measure equal to the value of equity’s comprehensive limited liability option divided by ongoing equity value \( -O_{t+} / E_{t+} \). The value of the comprehensive limited liability option
is a particularly meaningful indicator of financial risk as it combines consideration of both magnitude and probability of shortfall in satisfying due liabilities across all possible price and quantity outcomes for production output; dividing the option’s value by ongoing equity value provides a relative measure.

As further measures of financial risk, two probability of bankruptcy ($\hat{\Pr}_t B_{t-}$, $\hat{\Pr}_{t+} B_{t+}$) calculations and an output price beta ($\hat{\beta}_{t+}$) calculation are considered. $\hat{\Pr}_t B_{t-}$ is the probability of bankruptcy over the preceding production period due to production quantity risk. $\hat{\Pr}_{t+}$ and $\hat{\beta}_{t+}$ are risk-neutral measures of financial risk. $\hat{\Pr}_{t+}$ is the probability of bankruptcy for the ensuing production period. $\hat{\beta}_{t+}$ is the sensitivity of equity’s rate of return to the underlying output price rate of return for the ensuing production period (i.e. equity’s output price beta). $\hat{\beta}_{t+} > 1 (<1)$ implies that the return to equity is expected to be more (less) than one-for-one with the return to the underlying output price.

**Exogenous parameters**

To undertake a numerical analysis, the parameter values shown in Table 1 are chosen to represent the model company. Having already specified there to be three controlled production periods ($N = 3$), the time-span of each production period ($\Delta t$) is set to four years. The initial median periodic production quantity ($\bar{q}_0$) is standardised to 100 units, and commensurately the total output resource ($R$) for the three controlled production periods plus the uncontrolled fourth production period is assumed to be 400 units. Total production costs ($c$) are standardised to one dollar per unit of expected production quantity. The initial output price ($p_0$) is set to 1.5 dollars per unit; this represents an initial price margin of 50% over the expected unit production cost.

The risk-free interest rate ($r$) is set to 0.04 per year and output convenience yield ($\delta$) is set to 0.02 per year. Output price volatility ($\sigma_p$) is set to 0.2 per year and production quantity uncertainty ($\sigma_q$) is set to 0.1 per year. The correlation between the output price and production quantity ($\rho$) is set to a value of 0.1. It is intended that $\rho$ be representative of an endogenous operational strategy rather than an exogenous condition of the market for the company’s production output. That is, the company is assumed to be a price-taker, and
\( \rho \) parameterises a fixed strategy of shifting (expected) production quantity across time dependent on the output price level; presuming that production rate response to price change can occur concurrently with price change each production period.

<table>
<thead>
<tr>
<th>Table 1 – Exogenous parameter values</th>
</tr>
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<tbody>
<tr>
<td>Parameter values chosen to represent the model company.</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Exogenous parameter</th>
<th>Value</th>
<th>Exogenous parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Number of controlled production periods</td>
<td>( N )</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Production period time-span</td>
<td>( \Delta t )</td>
<td>4 years</td>
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</tr>
<tr>
<td>Total output resource</td>
<td>( R )</td>
<td>400 units</td>
<td></td>
</tr>
<tr>
<td>Initial output price</td>
<td>( p_0 )</td>
<td>$1.5/unit</td>
<td></td>
</tr>
<tr>
<td>Initial median periodic production</td>
<td>( \overline{q}_0 )</td>
<td>100 units</td>
<td></td>
</tr>
<tr>
<td>Output price volatility</td>
<td>( \sigma_p )</td>
<td>0.2/year</td>
<td></td>
</tr>
<tr>
<td>Production quantity uncertainty</td>
<td>( \sigma_q )</td>
<td>0.1/year</td>
<td></td>
</tr>
<tr>
<td>Output price and production quantity correlation</td>
<td>( \rho )</td>
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<td></td>
</tr>
<tr>
<td>Output convenience yield</td>
<td>( \delta )</td>
<td>0.02/year</td>
<td></td>
</tr>
<tr>
<td>Total production costs per unit of expected production quantity</td>
<td>( c )</td>
<td>$1/unit</td>
<td></td>
</tr>
<tr>
<td>Risk-free interest rate</td>
<td>( r )</td>
<td>0.04/year</td>
<td></td>
</tr>
<tr>
<td>Corporate tax rate</td>
<td>( \alpha )</td>
<td>0.35</td>
<td></td>
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<tr>
<td>Personal tax penalty on debt’s corporate tax shield rate</td>
<td>( \alpha - A_{\alpha} )</td>
<td>0.25</td>
<td></td>
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<tr>
<td>Claimability of a tax refund for a loss</td>
<td>( \lambda )</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Financial distress cost rate</td>
<td>( \gamma )</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Bankruptcy cost rate</td>
<td>( b )</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>Free cash-flow misappropriation rate</td>
<td>( a )</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Hedge transaction cost rate</td>
<td>( \varepsilon )</td>
<td>0.005</td>
<td></td>
</tr>
</tbody>
</table>

The corporate tax rate (\( \alpha \)) is set to the top earnings level US federal rate of 0.35 (state taxes are ignored). The claimability of a tax refund in event of negative earnings (\( \lambda \)) is set to 0.5 (to represent an ‘average’ tax shield benefit from carry-forward and carry-back provisions for losses). The personal tax penalty against debt’s role as a corporate tax shield is the corporate tax rate less the effective rate of combined personal and corporate tax being shielded by debt finance (\( \alpha - A_{\alpha} \)). Based on the results of Graham (1999), the value of \( A_{\alpha} \) is set to 0.1, hence (\( \alpha - A_{\alpha} \)) is set to 0.25. The rate of management misappropriation of positive free cash-flow (\( a \)) and the hedge transaction cost rate (\( \varepsilon \)) are each set to 0.005.

The financial distress cost rate (\( \gamma \)) is set to a value of 0.3. That is, the financial distress cost equals 30% of any shortfall in free cash-flow (\( -0.3F_t \) given \( F_t < 0 \)), up to the free cash-flow shortfall limit at which financial distress becomes bankruptcy, which entails a
maximum for financial distress cost equal to 23% of ongoing (post-distress) equity value \((-0.3F_i \leq 0.3E_{i+} \div (1+0.3))\). This maximum financial distress cost level is broadly congruent with the empirical evidence of Andrade and Kaplan (1998), who estimated the cost of financial distress to be of the order of 10% to 20% of pre-distress total company value (note, however, the difference in measurement basis).

The bankruptcy cost rate \((b)\) applies as a factor of debt face-value. By model set-up, with occurrence of bankruptcy, all due liabilities are honoured (if possible) with pre-tax liquidation cash-flow. Thus to avoid instance of abandonment/bankruptcy being favourably used to return capital to debt-holders from pre-tax cash-flow, \(b\) is set to a value greater than the corporate tax rate \((\alpha)\). Furthermore, the financial distress cost rate does not get applied with the occurrence of bankruptcy (although it does have an opportunity cost effect via the free cash-flow shortfall limit at which bankruptcy is instigated). Hence \(b\) is set to a value that is considered to reflect financial distress costs, not just direct bankruptcy costs. Considering again the order of magnitude of Andrade and Kaplan’s (1998) financial distress cost estimation relative to pre-distress total company value, \(b\) is set to a value of 0.4.

### 3. RESULTS

Figures 1a to 1h present the various optimal leverage, optimal hedge, total value and financial risk results for the model company at the start of each controlled production period, for each possible output price-path, and for each of the three control strategies, with the exogenous parameters set to the values provided in Table 1.

Figures 1a to 1h display the results at each controlled price-node of the non-recombining price-tree. At the end of the first controlled production period (at time \(t = \Delta t\)), there are five possible output price outcomes \((p_{\Delta} \in \{0.67, 1.01, 1.50, 2.24, 3.34\})\). These five possible values for \(p_{\Delta}\) each branch out to five possible price outcomes at the end of the second controlled production period \((p_{2\Delta})\). Thus there are five sets of five possible outcomes for \(p_{2\Delta}\) conditional on \(p_{\Delta}\). The sets of \(p_{2\Delta}\) conditional on \(p_{\Delta}\) are overlapping so that there are nine possible unconditional outcomes for \(p_{2\Delta}\) \((p_{2\Delta} \in \{0.30, 0.45, 0.67, 1.01, 1.50, 2.24, 3.34, 4.98, 7.43\})\). The model company is optimally abandoned/bankrupt
for price outcome $p_{\Delta t} = 0.67$ at end-of-period one, consequently Figures 1a to 1h omit display of the redundant set of five possible $p_{2\Delta t}$ outcomes conditional on $p_{\Delta t} = 0.67$ (i.e. Figures 1a to 1h display only four of the five sets of five possible outcomes for $p_{2\Delta t}$ conditional on $p_{\Delta t}$). At each price-node ($p_0$, $p_{\Delta t}$ and $p_{2\Delta t}$ conditional on $p_{\Delta t}$) the results for the three control strategies are displayed as a bar graph and also given numerically.

In reviewing the results, be aware that at each price-node there exists the risk of bankruptcy due to production quantity uncertainty for the preceding production period. For each time $t$, being the instant when a production period finishes and another begins, define time $t_+$ to occur instantaneously after time $t$. Given the control variables and knowing the output price ($p_t$) at a price-node, it is the production quantity outcome ($q_t$) that finally determines at time $t$ whether a condition of solvency, financial distress or bankruptcy is in effect and the consequential cash-flows to be distributed to stakeholders. Thus at time $t_+$ the company will be ongoing only if bankruptcy has not occurred at time $t$ (or at an earlier instance). The probability of such an ‘ex-post’ bankruptcy (i.e. the probability of bankruptcy due to quantity risk for the period preceding the price-node) is given by the calculation of $\Pr B_{t_+}$, for which the results presented in Figure 1g show only a few instances when quantity risk is relatively so large as to cause notable uncertainty about whether the company will be ongoing at a price-node. Furthermore, result measures given the subscript $t_+$ imply ongoing values, meaning that they are conditional on bankruptcy having not occurred at time $t$ (or earlier).

### 3.1. Leverage, hedging and company valuation

The optimal leverage results shown in Figure 1a indicate that optimal joint hedging and leverage almost always entails higher leverage than optimal leverage without hedging. Only in event of a very bullish price outcome ($p_{2\Delta t} = 7.43$ given $p_{\Delta t} = 3.34$, or $p_{2\Delta t} = 4.98$ given $p_{\Delta t} = 3.34$ or $p_{\Delta t} = 2.24$) does the levered, unhedged strategy entail higher leverage than the levered and hedged strategy. Nevertheless, from comparison with the optimal hedge results shown in Figure 1b, it is evident that the relationship between hedging and leverage cannot always be straightforwardly described as facilitative (from hedging to leverage).
The valuation benefit of joint hedging and leverage is observable in the differences between the values of $\hat{E}_{t-\Delta}[E_{t-}+(D_{t-}+O_{t-})|p_t]$ at time $t=0$ for each of the three control strategies as shown in Figure 1c. The $\hat{E}_{t-\Delta}[E_{0-}+O_{0-}|p_{0-}]$ measure (n.b. $D_{0-}=0$ by model set-up) is the all-equity value of the company prior to any control decisions being instigated, which provides a uniform valuation basis for comparison of the three control strategies. That is, $\hat{E}_{t-\Delta}[E_{0-}+O_{0-}|p_{0-}]$ is the present value of expected cash-flows to the initial all-equity owners of the company, and differences in $\hat{E}_{t-\Delta}[E_{0-}+O_{0-}|p_{0-}]$ arise purely due to the cash-flow effects of differences in control strategy going forward in time.

Figure 1c (at time $t=0$) shows that the levered and hedged strategy is more valuable than the levered, unhedged strategy by 3.4%. This is similar to the results of Leland’s (1998) modelling approach, which combined leverage optimisation with a very simple ‘hedging’ decision entailing a choice between high or low risk for the firm’s unlevered asset value. For various hedging risk levels, Leland’s jointly value-maximising hedging and leverage strategy (which he termed ‘ex-ante optimal’) provided a value benefit of between 2.1% and 5.6% relative to a strategy of value-maximising leverage with no hedging.

Specific to the interaction of hedging and leverage, the empirical evidence of Graham and Rogers (2002) indicated that firms are motivated to hedge to increase debt capacity and thereby access tax savings worth an estimated 1.1% of firm value on average. They also estimated hedging to be accountable for a 3% increase in debt-to-assets leverage on average, a result similar to the average leverage difference between non-financial firms with and without derivatives observed by Hentschel and Kothari (2001), but much more moderate than the leverage effect implied by the results of Leland (1998) and this study. The optimal debt-to-assets (debt-to-equity) leverage for Leland’s model firm increased from 43% (0.75) without hedging to between 52% (1.08) and 70% (2.33) with ex-ante optimal hedging and different hedging risk levels. For this study, the levered, unhedged strategy, with leverage $(\ell_{ts})$ ranging from zero to 1.68 depending on price-node, usually entails markedly lower leverage than the levered and hedged strategy, for which leverage ranges from 0.12 to 5.63 depending on price-node (see Figure 1a). Tufano (1996) also reported a large difference in leverage between unhedged gold miners and extensively hedged gold miners. Nevertheless the leverage values of the levered and hedged strategy tend to be high relative to empirically observed values, but are generally commensurate with the theoretical values of Leland’s (1998) ex-ante optimal strategy.
A prominent result observable in Figure 1a is that pecking order apparently dominates the model company’s leverage decision when made in conjunction with the hedging decision. For the levered and hedged strategy, leverage is clearly negatively related to output price outcome each period. Hedging initially ramps-up trade-off leverage, but the hedge loss after a price rise provides a tax shield substitute to debt, and the hedge profit after a price fall contributes motivation for tax-lowering leverage; hence trade-off and pecking order behaviours become somewhat aligned. The dampening effect of hedge losses on trade-off leverage is demonstrated by the result that the levered, unhedged strategy entails higher leverage than the levered and hedged strategy for strongly rising price outcomes $p_{2_{t}} = 7.43$ and $p_{2_{t}} = 4.98$ (with strongly rising price, trade-off leverage considerations will tend to have sway over pecking order).

The model company’s hedging behaviour is shown in Figure 1b. For the levered and hedged strategy, the model company establishes a term structure of short forwards at time $t=0$, but in response to a ‘weak’ output price outcome ($p_{t} = 1.01$, $p_{2_{t}} \leq 1.50$) the company actively reverses its overall hedge position in part or in whole. This corresponds with evidence from Graham and Rogers (2002) who found that firms reduce hedging in response to accumulated operating losses. In fact, if output price falls to near the expected unit production cost at commencement of the third controlled production period ($p_{2_{t}} = 1.01$), the hedge position is switched to a speculatively long stance. As such the hedge-delta ratio of the levered and hedged strategy varies from a minimum of -0.11 up to 0.54 depending on price-node. For comparison, Tufano’s (1996) sample of gold miners had an average (minimum, median, maximum) hedge-delta of 0.26 (zero, 0.23, 0.86).

By model set-up, together with every control decision, debt-holders and other non-equity stakeholders are fairly compensated for their limited liability risk. Hence the apparent asset substitution decision to reduce hedging or switch to speculation does not manifest wealth transfer from non-equity to equity. Instead it must be driven by cost/benefit asymmetry between upside and downside output price outcomes. At outset (time $t=0$) the model company’s jointly optimal hedging and leverage decision boosts the levels of both hedging and leverage (compared to the company being optimally hedged and unlevered,\(^9\) and

\(^9\) A value-maximising unlevered hedging strategy was also assessed, but the results are omitted for sake of brevity.
optimally levered and unhedged respectively). Then as long as output price is generally rising (from \( p_0 = 1.50 \) to \( p_{M} \geq 1.50 \) and \( p_{2M} \geq 2.24 \)), the hedging level is largely maintained, while leverage generally falls due to maturing-hedge losses substituting for debt as a tax shield. For conversely weak price outcomes (\( p_{M} = 1.01, \ p_{2M} \leq 1.50 \)), the company’s ongoing operational value is low, but the hedge portfolio is deep in-the-money and facilitative of increased leverage driven by pecking order roll-over of hedging-boosted debt plus tax-shielding of maturing-hedge profit. For the hedging decision, low ongoing operational value imposes a net cost/benefit asymmetry with respect to output price outcome, because an output price fall to below unit production cost will lead to abandonment/bankruptcy regardless of hedge portfolio value,\(^{10}\) but an output price rise will warrant continued operation and production.

Given a weak output price outcome and attendant deep in-the-money hedge portfolio, high leverage and low ongoing operational value, the operational benefit of a subsequent output price rise is vulnerable to the financial risk posed by the high leverage and the fact that the hedge portfolio would lose value; that is, an output price rise actually entails adversely high risk of bankruptcy and consequential foregone profitable production. To counter such circumstance, reduced hedging or even a switch of the hedging decision to a speculatively long stance increases the likelihood that the company will avoid bankruptcy in event of an output price rise and thereby benefit from ongoing profitable production. Consequently the facilitative relationship from hedging to leverage gets completely undone.

3.2. Financial risk

At outset the levered and hedged strategy has, by all risk measures considered, equal or lower financial risk than the other two control strategies (see Figures 1d to 1h at time \( t = 0 \)). But this changes depending on output price behaviour and risk measure. The relative limited liability risk measure (\( -O_{t} / E_{t} \), see Figure 1d) indicates the proportion of equity value (\( E_{t} \)) attributable to the value of equity’s comprehensive limited liability option (\( -O_{t} \)). The hedge-reducing behaviour associated with the levered and hedged

\(^{10}\) Abandonment allows the in-the-money hedge portfolio to be cashed-in instead of used to subsidise uneconomic production.
strategy at weak price outcomes \((p_{\Delta t} = 1.01, \quad p_{2\Delta t} \leq 1.50)\) exacerbates limited liability risk; sometimes this entails a value of \(\frac{-O_{t+}}{E_{t+}}\) greater than one, meaning that the limited liability option is in-the-money (i.e. unlimited liability equity would have negative value). As long as output price is generally rising (from \(p_0 = 1.50\) to \(p_{\Delta t} \geq 1.50\) and \(p_{2\Delta t} \geq 2.24\)), limited liability risk is very low or zero for all control strategies.

Equity’s output price beta (\(\hat{\beta}_{t+}\), see Figure 1f) varies between 1.14 and 4.23 across all output price outcomes and control strategies, and is clearly negatively related to output price outcome. Nearly always the levered and hedged strategy has the lowest beta of the three control strategies, except in event of a very weak price outcome \((p_{2\Delta t} = 1.01)\) when the hedging decision turns speculative. These results are consistent with the empirical evidence of Tufano (1998), whose sample of gold miners had a mean (first quartile, median, third quartile) three-monthly gold price beta equal to 2.21 (1.13, 2.09, 3.13). Tufano found the miners’ gold price betas to be negatively related to their extent of hedging, positively related to their financial leverage, and negatively related to the gold price level.

The ex-post probability of bankruptcy measure (\(Pr B_{t-}\), see Figure 1g) indicates the probability of bankruptcy at a price-node due to production quantity risk for the preceding production period. For the first two controlled production periods, production quantity risk predominantly poses negligible bankruptcy threat, though distinct exception applies for the unlevered, unhedged and levered, unhedged strategies at low price outcome \(p_1 = 1.01\).

Figure 1h shows that the risk-neutral ongoing (unconditional) probability of bankruptcy (\(\hat{Pr} B_{t+}\)) for the levered and hedged strategy only becomes large in comparison to the other two control strategies for weak price outcomes commencing the third controlled production period \((p_{2\Delta t} \leq 1.50)\). That is, although the levered and hedged strategy demonstrates hedge reduction for a weak price outcome commencing the second controlled production period \((p_{\Delta t} = 1.01)\), the likelihood of the model company surviving into the third production period is not disadvantaged relative to the other control strategies. This result arises from the fact that, in response to a weak price outcome at time \(t = \Delta t\), the company uses its hedging term structure to both promote survival (since the company is early in its production life), and to pre-empt a reduced hedging position if price weakness persists at time \(t = 2\Delta t\) leading into the third and penultimate production period.
Figure 1a – Price-path-dependent optimal leverage

Optimal (value-maximising) leverage \( \ell_{t,r} = D_{t,r} / E_{t,r} \) for the model company at the start of each controlled production period, for each possible output price-path, and for each of three control strategies, with exogenous parameters set to values provided in Table 1.

Figure 1b – Price-path-dependent optimal hedge-delta ratio

Optimal (value-maximising) hedge-delta ratio \( h_{t,r} \) for the model company at the start of each controlled production period, for each possible output price-path, and for each of three control strategies, with exogenous parameters set to values provided in Table 1.
Figure 1c – Price-path-dependent total company value (pre-control decisions)

Total value of the model company at the start of each controlled production period instantaneously before new control decisions are instigated \((\hat{E}_{t-r} [E_{t-r} + (D_{t-r} + O_{t-r}) | p_t])\), for each possible output price-path, and for each of three control strategies, with exogenous parameters set to values provided in Table 1.

Figure 1d – Price-path-dependent limited liability risk

Relative limited liability risk \((-O_{rs} / E_{rs})\) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of three control strategies, with exogenous parameters set to values provided in Table 1.
Figure 1f – Price-path-dependent beta

Equity’s output price beta ($\beta_{t}$) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of three control strategies, with exogenous parameters set to values provided in Table 1.

Figure 1g – Price-path-dependent ex-post probability of bankruptcy

Probability of bankruptcy for the preceding production period due to production quantity risk ($Pr_{t}$) for the value-maximised model company at the start of each controlled production period, for each possible output price-path, and for each of three control strategies, with exogenous parameters set to values provided in Table 1.
4. CONCLUSION

To investigate both the value and risk impacts of a joint hedging and leverage decision, this study utilises a multi-period model for a company subject to respectively hedgeable and unhedgeable production output price and quantity risk variables, endogenously derived deadweight costs, and the tandem availability of risky leverage and flexible hedging control variables. It is found that an optimal (value-maximising) joint hedging and leverage strategy can increase company value by about 3.4% compared to an unhedged optimal leverage strategy and by about 3.6% compared to the company being unlevered and unhedged. Also found is that optimal leverage is usually much higher in conjunction with optimal hedging than with no hedging, but the relationship is observed to not purely be a matter of higher hedging facilitating higher leverage.

At outset a jointly optimal hedging and leverage strategy markedly boosts the levels of both hedging and leverage, compared to optimal hedging without leverage, and optimal
leverage without hedging respectively. Then as long as the price for production output is at least rising modestly, the hedging level is largely maintained, while leverage tends to fall due to maturing-hedge losses substituting for debt as a tax shield. For conversely weak output price outcomes close to the unit cost of production, the ongoing operational value of the company is low, particularly if the company is at a late stage of its production life, but the hedge portfolio is deep in-the-money and facilitative of increased leverage driven by pecking order roll-over of hedging-boosted debt plus tax-shielding of maturing-hedge profit. For the hedging decision, low ongoing operational value imposes a net cost/benefit asymmetry with respect to output price outcome, because an output price fall (below unit production cost) will lead to abandonment/bankruptcy regardless of hedge portfolio value, but an output price rise will warrant continued operation and production. Given a weak price outcome and attendant deep in-the-money hedge portfolio, high leverage and low ongoing operational value, the operational benefit of a subsequent output price rise is vulnerable to the financial risk posed by the high leverage and the fact that the hedge portfolio would lose value; that is, an output price rise actually entails adversely high risk of bankruptcy and consequential foregone profitable production. To counter such circumstance, reduced hedging or even a switch of the hedging decision to a speculatively long stance increases the likelihood that the company will avoid bankruptcy in event of an output price rise and thereby benefit from ongoing profitable production. The value-maximising optimality of this risk-seeking behaviour arises in conjunction with fair compensation to debt-holders and other non-equity stakeholders for limited liability risk.

Measures of financial risk consequently demonstrate that, for weak output price outcomes close to the unit cost of production, an optimal joint hedging and leverage strategy is often more risky than either an unhedged optimal leverage strategy or an unhedged and unlevered strategy; but the opposite is often evident at higher price levels. Nevertheless, overall distinction of the relative financial riskiness of optimal joint hedging and leverage versus alternative strategies is not easily generalised and shows dependence not only on the output price level, but also on the company’s remaining production life.

Further work is required to determine whether the general results of this study qualitatively persist when the exogenous parameters are varied between reasonable extremes.
APPENDIX A. THE MODEL

The intention is to define the model company’s equity and non-equity\textsuperscript{11} stakes as complex, interdependent, American-style options on production and hedging cash-flows. Thereby an option-pricing approach can be used to value equity and non-equity and to also assess the effects of hedging and leverage control decisions. The model company faces two sources of uncertainty: at time $t$, being the end of a discrete production period of duration $\Delta t$ years, the production output for the period is an uncertain quantity of $q_t$ units, and the price obtainable for the production output is an uncertain amount of $p_t$ dollars per unit. The bivariate price and quantity process is assumed to be Markovian. Hedging can only be contracted with respect to price uncertainty (i.e. the price risk is hedgeable and the quantity risk is unhedgeable). The company is assumed to have uniform production periods, and periodic expected production quantity is assumed to be independent of previous unexpected production quantity deviations. This allows the company to be specified as blind to the history of the stochastic component of production quantities, which advantageously allows equity and non-equity, conditional on price and control variables, to have generalisable valuation solutions with respect to quantity uncertainty.

Price and quantity random variable specification

Assume that the production output price and quantity are lognormal random variables such that their risk-neutral processes are:

\[
\begin{bmatrix}
\ln p_t \\
\ln q_t \\
\end{bmatrix} = \\
\begin{bmatrix}
\ln p_{t-\Delta t} + \left( r - \delta - \frac{\sigma_p^2}{2} \right) \Delta t \\
\ln \bar{d}_0 + \rho \left( \frac{\sigma_q}{\sigma_p} \right) \ln \left( \frac{p_{t-\Delta t}}{p_0} \right) - \left( r - \delta - \frac{\sigma_p^2}{2} \right) (t - \Delta t) \\
\end{bmatrix} + \begin{bmatrix}
\sigma_p \psi_t^{(p)} \sqrt{\Delta t} \\
\sigma_q \psi_t^{(q)} \sqrt{\Delta t} \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
\psi_t^{(p)} \\
\psi_t^{(q)} \\
\end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)
\]

\textsuperscript{11} The non-equity stake refers to the combined position of debt plus the short comprehensive limited liability option provided to equity by all non-equity stakeholders in the company.
where: \( r \) is the annual, continuously compounding risk-free interest rate; \( \delta \) is the annual, continuously compounding convenience yield of the production output; \( p_0 \) and \( q_0 \) are respectively the initial output price and the initial median periodic production quantity; \( \sigma_p^2 \) and \( \sigma_q^2 \) are the annual variance rates for the output price and production quantity respectively; and \( \rho \) is the correlation between output price and production quantity.\(^{12}\)

For the sake of model implementation, the periodic lognormal price distribution is approximated from a binomial process. The control decisions and price risk can then be incorporated into the valuation problem via a price-tree, specifically with non-recombining nodes so as to enable path-dependent control behaviour. Define \((n+1)\) to be the number of possible future price realisations emanating from a price-node for a single production period (i.e. price realisations numbered from 0 to \( n \)). Hence the risk-neutral process for \( p_t \) is approximated by:

\[
p_t = p_{t-\Delta t} \left( e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)+\sigma_p \sqrt{\Delta t/n}} \right)^{n-P} \left( e^{\left(r-\delta-\sigma_q^2/2\right)(\Delta t/n)-\sigma_q \sqrt{\Delta t/n}} \right)^{(n-P)},
\]

\[
P \sim \text{BIN}\left(n, \frac{e^{\left(r-\delta-\sigma_p^2/2\right)(\Delta t/n)+\sigma_p \sqrt{\Delta t/n}}-e^{\left(r-\delta-\sigma_q^2/2\right)(\Delta t/n)-\sigma_q \sqrt{\Delta t/n}}}{e^{\left(r-\delta-\sigma_q^2/2\right)(\Delta t/n)+\sigma_q \sqrt{\Delta t/n}}-e^{\left(r-\delta-\sigma_q^2/2\right)(\Delta t/n)-\sigma_q \sqrt{\Delta t/n}}\right).
\]

The conditional distribution of \( q_t \) is:

\[
\left(\ln q_t \mid p_t\right)_{t-\Delta t} \sim N\left(\ln q_0 + \rho \left(\frac{\sigma_q}{\sigma_p}\right) \ln \left(\frac{p_t}{p_0}\right) - \left(r - \delta - \frac{\sigma_p^2}{2}\right) t, \sigma_q^2 (1 - \rho^2) \Delta t\right)
\]

and the conditional expectation of \( q_t \) is:

\[
\hat{E}_{t-\Delta t}[q_t \mid p_t] = \bar{q}_0 \left(\frac{p_t}{p_0}\right) \exp \left\{ -\rho \left(\frac{\sigma_q}{\sigma_p}\right) \left( r - \delta - \frac{\sigma_p^2}{2}\right) t + (1 - \rho^2) \frac{\sigma_q^2}{2} \Delta t \right\}
\]

where \( \hat{E}_{t-\Delta t}[\cdot] \) is the risk-neutral expectation operator.

\(^{12}\) A positive correlation can be used to represent a price-taking company that adjusts expected production to follow step with price movements.
Furthermore, for later reference, the risk-neutral conditional expected payoffs for a vanilla put option, an asset-or-nothing digital put option and a cash-or-nothing digital put option contracted on $q_t$ with strike $\chi$ (respectively designated as $VP_{q\chi}$, $ADP_{q\chi}$ and $CDP_{q\chi}$) are:

$$VP_{q\chi} = \hat{E}_{t-M}[(\chi - q_t) I_{q\chi} | p_t]$$

$$= \chi \Phi \left( \frac{\ln \chi - \hat{E}_{t-M} [\ln q_t | p_t]}{\sigma \sqrt{(1 - \rho^2) \Delta t}} \right) - \hat{E}_{t-M} [q_t | p_t] \Phi \left( \frac{\ln \chi - \hat{E}_{t-M} [\ln q_t | p_t] - \sigma^2 (1 - \rho^2) \Delta t}{\sigma \sqrt{(1 - \rho^2) \Delta t}} \right),$$

$$ADP_{q\chi} = \hat{E}_{t-M} [q_t I_{q\chi} | p_t] = \hat{E}_{t-M} [q_t | p_t] \Phi \left( \frac{\ln \chi - \hat{E}_{t-M} [\ln q_t | p_t] - \sigma^2 (1 - \rho^2) \Delta t}{\sigma \sqrt{(1 - \rho^2) \Delta t}} \right),$$

$$CDP_{q\chi} = \hat{E}_{t-M} [I_{q\chi} | p_t] = \Phi \left( \frac{\ln \chi - \hat{E}_{t-M} [\ln q_t | p_t]}{\sigma \sqrt{(1 - \rho^2) \Delta t}} \right),$$

where: the indicator function, $I_{\text{logical statement}}$, equals one if the logical statement is true and zero otherwise; and $\Phi \{ \cdot \}$ is the standard normal cumulative distribution function.

**Control variables**

Define positive integer $N$ to be the number of production periods for which control decisions will be made: $N$ may cover the company’s total production life or, if an estimate can be made for ongoing company value at some intermediate stage of its total life, $N$ may then be the number of production periods up to this intermediate stage. Time is denominated by the discrete production periods $\{ t \in \{0, \Delta t, 2\Delta t, ..., N\Delta t\} \}$. Given the positive integer approximation of $(n+1)$ possible price realisations for a production period, the total number of non-recombining price-paths is therefore equal to $(n+1)^N$. Define $\eta \in \{1, 2, ..., (n+1)^N\}$ to signify a specific price-path.

The company has control over hedging and leverage. The hedging control decision at each start-of-period entails application of a hedging term structure comprising a series of hedge quantities ($x_{t, t + k\Delta t}$ production units) for hedge contracts with initial terms to maturity ($k\Delta t$)
increasing from $\Delta t$ years (i.e. one period) to a maximum of $(N\Delta t - t)$ years (i.e. $(N\Delta t - t)/\Delta t$ periods); similarly the leverage control decision at each start-of-period entails application of a debt term structure in the form of a series of zero-coupon bonds designated by their dollar face-values ($y_{t,t+\kappa\Delta t} \geq 0$) also with initial terms to maturity ($\kappa\Delta t$) increasing from $\Delta t$ years to $(N\Delta t - t)$ years. For a given price-path ($\eta$), the entire matrices of hedging and leverage control variables are respectively given by:

$$
x_{\eta} = \begin{bmatrix}
x_{0,\Delta t} & 0 & 0 & \cdots & 0 \\
x_{0,2\Delta t} & x_{\Delta t,2\Delta t} & 0 & \cdots & 0 \\
x_{0,3\Delta t} & x_{\Delta t,3\Delta t} & x_{2\Delta t,3\Delta t} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
 x_{0,N\Delta t} & x_{\Delta t,N\Delta t} & x_{2\Delta t,N\Delta t} & \cdots & x_{(N-1)\Delta t,N\Delta t}
\end{bmatrix},
$$

$$
y_{\eta} = \begin{bmatrix}
y_{0,\Delta t} & 0 & 0 & \cdots & 0 \\
y_{0,2\Delta t} & y_{\Delta t,2\Delta t} & 0 & \cdots & 0 \\
y_{0,3\Delta t} & y_{\Delta t,3\Delta t} & y_{2\Delta t,3\Delta t} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_{0,N\Delta t} & y_{\Delta t,N\Delta t} & y_{2\Delta t,N\Delta t} & \cdots & y_{(N-1)\Delta t,N\Delta t}
\end{bmatrix}.
$$

The columns of $x_{\eta}$ and $y_{\eta}$ represent each start-of-period’s hedging and leverage choices for $t = 0$ to $t = (N-1)\Delta t$. With each increment in time, the number of remaining production periods reduces by one, and accordingly the number of term structure elements in each of the hedging and leverage control decisions also reduces by one. The number of non-recombining price-nodes, for each of which a unique hedging and leverage control decision can be made, amounts to $(n+1)^j/\Delta t$ at each new start-of-period from $t = 0$ to $t = (N-1)\Delta t$. Hence the total number of hedging and leverage elements that can potentially be chosen uniquely amounts to:

$$
2 \sum_{i=0}^{N-1} (n+1)^i (N-i).
$$

Obviously this can easily become a monumental task to undertake in a meaningful way, thus in application some simplifying restrictions will be made.

Now define $X_t$ to be the matrix of net values ($X_{t,t+\kappa\Delta t}$ in dollars per production unit) of ‘live’ and maturing hedge contracts; $X_{t,t+\kappa\Delta t}$ is undefined (set to zero) if specified time $t$...
is earlier than hedge contract initiation at time \( \tau \) or later than hedge contract maturity at time \( (\tau + \kappa \Delta t) \) (i.e. \( X_{t, \tau + \kappa \Delta t} = 0 \) if \( t < \tau \) or \( t > (\tau + \kappa \Delta t) \)):

\[
X_t = \begin{bmatrix}
X_{t_0, \Delta t} & 0 & 0 & \cdots & 0 \\
x_{t_0, 2\Delta t} & x_{\Delta t, 2\Delta t} & 0 & \cdots & 0 \\
x_{t_0, 3\Delta t} & x_{\Delta t, 3\Delta t} & x_{2\Delta t, 3\Delta t} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_{t_0, N\Delta t} & x_{\Delta t, N\Delta t} & x_{2\Delta t, N\Delta t} & \cdots & x_{(N-1)\Delta t, N\Delta t}
\end{bmatrix}.
\]

The hedge contracts are assumed to have zero upfront cost (i.e. the applicable upfront hedge contract premium, if any, is compounded forward at the risk-free interest rate and included as a negative amount in the net hedge payoff calculation at contract maturity). Hence the net value of each hedge contract at initiation \( (X_{t_2, t + \kappa \Delta t}) \) is set to zero. If the hedge contracts mature in favour of the company \( (X_{t_2 - \kappa \Delta t, t} > 0) \), they contribute to end-of-period revenue. If the hedge contracts mature out of the company’s favour \( (X_{t_2 - \kappa \Delta t, t} < 0) \), they become an end-of-period liability.

### Hedge specification

The model company is assumed able to combine a short forward hedge with a put option hedge. In addition to control of hedge quantity (described by matrix \( x_q \)), at each start-of-period the company can also choose a term structure for: the fraction of total hedge quantity committed to short forwards as opposed to put options \( (0 \leq w_{t, \tau + \kappa \Delta t} \leq 1) \), i.e. \( w_{t_2, t + \kappa \Delta t} x_{t_2, t + \kappa \Delta t} \) production units are hedged with short forwards and \( (1 - w_{t_2, t + \kappa \Delta t}) x_{t_2, t + \kappa \Delta t} \) units are hedged with put options; and the strike price of the put options \( (z_{t_2, t + \kappa \Delta t} > 0) \). The entire matrices of these additional hedging control variables are:

\[
w_q = \begin{bmatrix}
w_{0, \Delta t} & 0 & 0 & \cdots & 0 \\
w_{0, 2\Delta t} & w_{\Delta t, 2\Delta t} & 0 & \cdots & 0 \\
w_{0, 3\Delta t} & w_{\Delta t, 3\Delta t} & w_{2\Delta t, 3\Delta t} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
w_{0, N\Delta t} & w_{\Delta t, N\Delta t} & w_{2\Delta t, N\Delta t} & \cdots & w_{(N-1)\Delta t, N\Delta t}
\end{bmatrix}.
\]
The net hedge payoff of any particular hedge term structure element is given by:

\[
x_{t,\tau\kappa} X_{t,\tau\kappa} = x_{t,\kappa\tau} w_{t,\kappa\tau} \left[ (1-\varepsilon) p_{t,\kappa\tau} \kappa^{(r-\delta)\kappa\tau} - p_{t,\kappa\tau} \right] \\
+ x_{t,\kappa\tau} (1-w_{t,\kappa\tau}) \max \left[ 0, z_{t,\kappa\tau} - p_{t,\kappa\tau} \right] \\
- x_{t,\kappa\tau} (1-w_{t,\kappa\tau}) \left( \hat{E}_{t,\kappa\tau} \left[ \max \left[ 0, z_{t,\kappa\tau} - p_{t,\kappa\tau} \right] \right] + \varepsilon z_{t,\kappa\tau} \right)
\]

where \(0 \leq |\varepsilon| < 1\) is the transaction cost rate for the hedge contracts. The transaction cost rate should be positive \((0 \leq \varepsilon < 1)\) for ‘positive’-hedge positions (i.e. combined short forward and long put positions meaning that \(x_{t,\kappa\tau} > 0\)), and negative \((-1 < \varepsilon \leq 0)\) for ‘negative/anti’-hedge positions (i.e. combined long forward and short put positions meaning that \(x_{t,\kappa\tau} < 0\)). The hedge transaction cost rate is charged against the forward price for forward contracts and, for equivalent scale effect, against the strike price for put options. Note that the put option premium is deferred for payment at hedge maturity and incorporated into the net hedge payoff as previously specified.

The value of a live hedge position \(\tau < t < \tau + \kappa\Delta t\) is given by:

\[
x_{t,\tau\kappa} X_{t,\tau\kappa} = e^{-\tau(\tau+\kappa\Delta t)} x_{t,\tau\kappa} w_{t,\tau\kappa} \left[ (1-\varepsilon) p_{t,\tau\kappa} e^{(r-\delta)\kappa\tau} - p_{t,\tau\kappa} \right] \\
+ e^{-\tau(\tau+\kappa\Delta t)} x_{t,\tau\kappa} (1-w_{t,\tau\kappa}) \hat{E}_{t,\tau\kappa} \left[ \max \left[ 0, z_{t,\tau\kappa} - p_{t,\tau\kappa} \right] \right] \\
- e^{-\tau(\tau+\kappa\Delta t)} x_{t,\tau\kappa} (1-w_{t,\tau\kappa}) \left( \hat{E}_{t,\tau\kappa} \left[ \max \left[ 0, z_{t,\tau\kappa} - p_{t,\tau\kappa} \right] \right] + \varepsilon z_{t,\tau\kappa} \right).
\]

The hedge transaction cost is due at hedge maturity, and the liability for the hedge transaction cost is assumed to not come into effect until instantaneously after hedge initiation, therefore precisely at initiation \((t = \tau)\) the hedge position has zero value as previously specified (i.e. \(x_{t,\tau\kappa} X_{t,\tau\kappa} = 0\)).
Operating earnings and cash-flows

Define \( c \geq 0 \) as the total costs of production in dollars per unit of expected production conditional on price outcome. Total production costs \( (c \hat{E}_{t-\Delta t}[q_t \mid p_t]) \) thereby vary with the expected production quantity response to price outcome, but do not vary with that part of quantity risk that is orthogonal to price outcome. The intention is that the model company is able to have a predictable production response to price behaviour as specified by the correlation coefficient \( (\rho) \), and that total production costs adjust for this predictable response but are otherwise fixed.

Earnings before interest and tax \( (EBIT_t) \) is comprised of operating profit \( (p_t q_t - c \hat{E}_{t-\Delta t}[q_t \mid p_t]) \), plus total net payoff of maturing hedge contracts:

\[
EBIT_t = p_t q_t - c \hat{E}_{t-\Delta t}[q_t \mid p_t] + (e_t^T (x_t)) (e_t^T X_t)^T
\]

where \( e_t \) is an array of length \( N \) made up of zeros except for a one at the \( t/\Delta t \) array position, and the superscript \( T \) signifies an array or matrix transpose. Free cash-flow before tax \( (FBT_t) \) comprises \( EBIT_t \), plus proceeds of new bond issues \( (Y_t) \), minus payment of face-values of maturing bonds:

\[
FBT_t = p_t q_t - c \hat{E}_{t-\Delta t}[q_t \mid p_t] + (e_t^T (x_t)) (e_t^T X_t)^T + Y_t - (e_t^T y_n) 1
\]

where \( 1 \) is an array of length \( N \) made up of ones. Introducing the function \( f(p_t) \) to represent total due pre-tax liabilities \( (c \hat{E}_{t-\Delta t}[q_t \mid p_t] + (e_t^T y_n) 1) \), less net hedge payoff \( (e_t^T (x_t)) (e_t^T X_t)^T \), less new debt finance \( (Y_t) \), then \( FBT_t \) can be rewritten:

\[
FBT_t = p_t q_t - \left[ c \hat{E}_{t-\Delta t}[q_t \mid p_t] + (e_t^T y_n) 1 - (e_t^T (x_t)) (e_t^T X_t)^T - Y_t \right] = p_t q_t - f(p_t).
\]

It is assumed that the interest expense of each newly issued bond (i.e. face-value minus initial value) can be immediately brought to account against \( EBIT_t \). Hence earnings before tax \( (EBT_t) \) is given by:
\[ EBT_t = p_t q_t - c \hat{E}_{t+\Delta} [q_t | p_t] + (\epsilon_t^T x_{\eta_t})^T X_t - \left[ \mathbf{1}^T (y_{\eta_t} e_{t+\Delta}) - Y_t \right] \]

\[ = FBT_t + (\epsilon_t^T y_{\eta_t}) \mathbf{1} - \mathbf{1}^T (y_{\eta_t} e_{t+\Delta}) \]

\[ = p_t q_t - f(p_t) - \left[ \mathbf{1}^T (y_{\eta_t} e_{t+\Delta}) - (\epsilon_t^T y_{\eta_t}) \mathbf{1} \right]. \]

Positive \( EBT_t \) is assumed subject to a tax rate of \( 0 \leq \alpha < 1 \), while negative \( EBT_t \) is assumed subject to a tax rate of \( \lambda \alpha \), where \( 0 \leq \lambda \leq 1 \) represents the claimability of a tax refund in event of a negative \( EBT_t \) result.\(^{13}\) Therefore free cash-flow \( (F_t) \) comprises \( FBT_t \) minus (plus) tax payable (refundable) on positive (negative) \( EBT_t \). Define the function \( d_t(y_{\eta_t}) \) to equal the increment in total debt face-value, \( \left[ \mathbf{1}^T (y_{\eta_t} e_{t+\Delta}) - (\epsilon_t^T y_{\eta_t}) \mathbf{1} \right] \) (i.e. total face-value of newly issued bonds minus total face-value of maturing bonds). The expression for \( F_t \) is:

\[
F_t = FBT_t - \alpha EBT_t + (1 - \lambda) \alpha EBT_t I_{EBT_t < 0} \\
= FBT_t - \alpha \left\{ FBT_t - \left[ \mathbf{1}^T (y_{\eta_t} e_{t+\Delta}) - (\epsilon_t^T y_{\eta_t}) \mathbf{1} \right] \right\} I_{\left[ FBT_t \right]_{\left( y_{\eta_t} e_{t+\Delta} \right)} > \left( \epsilon_t^T y_{\eta_t} \right) t} \\
\quad + (1 - \lambda) \alpha \left\{ FBT_t - \left[ \mathbf{1}^T (y_{\eta_t} e_{t+\Delta}) - (\epsilon_t^T y_{\eta_t}) \mathbf{1} \right] \right\} I_{\left[ FBT_t \right]_{\left( y_{\eta_t} e_{t+\Delta} \right)} < \left( \epsilon_t^T y_{\eta_t} \right) t} \\
= (1 - \alpha) \left[ p_t q_t - f(p_t) \right] + \alpha d_t(y_{\eta_t}) + (1 - \lambda) \alpha \left\{ p_t q_t - f(p_t) - d_t(y_{\eta_t}) \right\} I_{\left[ \frac{d_t(y_{\eta_t})}{p_t} \right]_{(p_t) \left( y_{\eta_t} \right)}}. \\
\]

**Cash-flows to equity**

Solvency is defined to be the condition where free cash-flow is zero or positive \( (F_t \geq 0) \). It is assumed that a fraction \( (0 \leq \alpha \leq 1) \) of positive free cash-flow is misappropriated by management and the remainder is paid out as a dividend to equity. Financial distress is defined to be the condition of a free cash-flow shortfall \( (F_t < 0) \) necessitating new equity finance via a ‘negative dividend’ (which is considered to be a rights issue); an additional

\(^{13}\) The model does not specifically allow for the carry-forward or carry-back of tax losses. Instead the parameter \( \lambda \) is used to represent the company’s ‘average’ ability to claim a tax refund in event of a loss. A high value (i.e. \( \lambda \approx 1 \)) would be appropriate for a normally profitable company with previous profits that the loss can be carried-back against, or with high expectation of future profits that the loss can be carried-forward against.
fraction \((\gamma \geq 0)\) of the free cash-flow shortfall is required from equity finance providers to cover financial distress costs and the transaction costs of the equity issue. Bankruptcy is defined to occur under condition of negative free cash-flow such that the required equity-raising \((-1 + \gamma)F_i\) is larger than the ongoing equity value (being the present value of future cash-flows to equity given that bankruptcy has not occurred). Using \(E_t \geq 0\) to signify the ongoing equity value, the dividend payment \((G_t)\) is determined as follows:

\[
G_t = \begin{cases} 
(1-a)F_i, & F_i \geq 0 \\
(1 + \gamma)F_i, & -E_t, (1 + \gamma) \leq F_i < 0 \\
0, & F_i < -E_t, (1 + \gamma) 
\end{cases}
\]

\[
= (1-a)F_i + (a+\gamma)F_i I_{F_i \leq -(1+\gamma)E_t} - (1+\gamma)F_i I_{F_i < -(1+\gamma)E_t}.
\]

Noting, for some value \(J\), that:

\[
I_{F_i < J} = I_{\frac{(1-\alpha) f(p_i) + J - \alpha d_i(y_{\eta})}{(1-\alpha)p_i}} - I_{\frac{(1-\alpha) f(p_i) + J - \alpha d_i(y_{\eta})}{(1-\alpha)p_i}} + I_{\frac{(1-\alpha) f(p_i) + J - \alpha d_i(y_{\eta})}{(1-\alpha)p_i}}.
\]

Then substituting for \(F_i\) in the expression for \(G_t\) gives:

\[
G_t = (1-a)(1-\alpha)\left[p_t q_t - f(p_t)\right] + (1-a)\alpha d_i(y_{\eta})
\]

\[
-(1-a)(1-\lambda)\alpha p_t \left\{ \frac{f(p_t) + d_i(y_{\eta})}{p_t} - q_t \right\} I_{\frac{f(p_t) + d_i(y_{\eta})}{p_t}}
\]

\[
-(a+\gamma)(1-\alpha) p_t \left\{ \frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t} - q_t \right\} I_{\frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t}}
\]

\[
+(1+\gamma)(1-\alpha) p_t \left\{ \frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t} - q_t \right\} I_{\frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t}}
\]

\[
-(a+\gamma)(1-\alpha) p_t \left\{ \frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t} - q_t \right\} I_{\frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t}}
\]

\[
+(1+\gamma)(1-\alpha) p_t \left\{ \frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t} - q_t \right\} I_{\frac{(1-\alpha) f(p_t) - \alpha d_i(y_{\eta})}{(1-\alpha)p_t}}
\]

\[
-(a+\gamma)(1-\alpha\lambda) p_t \left\{ \frac{(1-\alpha\lambda) f(p_t) - \alpha\lambda d_i(y_{\eta})}{(1-\alpha\lambda)p_t} - q_t \right\} I_{\frac{(1-\alpha\lambda) f(p_t) - \alpha\lambda d_i(y_{\eta})}{(1-\alpha\lambda)p_t}}
\]

\[
+(1+\gamma)(1-\alpha\lambda) p_t \left\{ \frac{(1-\alpha\lambda) f(p_t) - \alpha\lambda d_i(y_{\eta})}{(1-\alpha\lambda)p_t} - q_t \right\} I_{\frac{(1-\alpha\lambda) f(p_t) - \alpha\lambda d_i(y_{\eta})}{(1-\alpha\lambda)p_t}}
\]
Note that \( f(p_t), d_t(y_q) \) and \( E_t, r \) are dependent on \( p_t \) but not \( q_t \), although \( q_t \) does have a role in determining whether bankruptcy occurs and thus whether \( E_t, r \) is accessed or not. This means that at each period the expected dividend payment can be solved conditional on \( p_t \). To do so it can be seen that the dividend payment comprises a series of put option payoffs dependent on \( q_t \): two standard/vanilla put options; five asset-or-nothing digital put options; and five cash-or-nothing digital put options. Recall that \( VP_{q,<z}, ADP_{q,<z} \) and \( CDP_{q,<z} \) signify the risk-neutral conditional expected payoffs respectively for a vanilla put option, an asset-or-nothing digital put option and a cash-or-nothing digital put option contracted on \( q_t \) with strike \( x \). To simplify notation, define \( x_1, x_2, x_3, x_4 \) and \( x_5 \):

\[
\begin{align*}
x_1 &= \frac{f(p_t) + d_t(y_q)}{p_t}, \quad x_2 = \frac{(1-\alpha)f(p_t) - \alpha d_t(y_q)}{(1-\alpha)p_t}, \\
x_3 &= \frac{(1+\gamma)(1-\alpha)f(p_t) - (E_{t,r} \mid p_t) - \alpha(1+\gamma)d_t(y_q)}{(1+\gamma)(1-\alpha)p_t}, \\
x_4 &= \frac{(1-\alpha\lambda)f(p_t) - \alpha\lambda d_t(y_q)}{(1-\alpha\lambda)p_t}, \\
x_5 &= \frac{(1+\gamma)(1-\alpha\lambda)f(p_t) - (E_{t,r} \mid p_t) - \alpha\lambda(1+\gamma)d_t(y_q)}{(1+\gamma)(1-\alpha\lambda)p_t}.
\end{align*}
\]

Hence:

\[
\hat{E}_{t,\Delta}[G_t \mid p_t] = \left[(1-a)(1-\alpha)\left[p_t \hat{E}_{t,\Delta}[q_t \mid p_t] - f(p_t)\right] + (1-a)\alpha d_t(y_q)\right] \\
- (1-a)(1-\lambda)\alpha p_t VP_{q,<z} -(a+\gamma)(1-\alpha)p_t \left\{VP_{q,<z} - \chi_2 CDP_{q,<\min[x_2,z]} + ADP_{q,<\min[x_2,z]} \right\} \\
- (a+\gamma)(1-\alpha\lambda)\alpha p_t \left\{X_4 CDP_{q,<\min[x_2,z]} - ADP_{q,<\min[x_2,z]} \right\} \\
+ (1+\gamma)(1-\alpha)p_t \left\{X_2 CDP_{q,<x_2} - ADP_{q,<x_2} - \chi_2 CDP_{q,<\min[x_2,z]} + ADP_{q,<\min[x_2,z]} \right\} \\
+ (1+\gamma)(1-\alpha\lambda)p_t \left\{X_4 CDP_{q,<\min[x_2,z]} - ADP_{q,<\min[x_2,z]} \right\}.
\]

In the event of bankruptcy \( (F_t < -E_{t,r} / (1+\gamma)) \), the company is liquidated for a cash-flow \( (L_t) \) equal to \( EBIT_t \) plus the market value of the company’s live (non-maturing) hedge contracts, minus the total face-value of outstanding debt (excluding new debt, which would not be issued in event of bankruptcy) increased by a bankruptcy cost factor \( (b) \):

\[
L_t = p_t q_t - e \hat{E}_{t,\Delta}[q_t \mid p_t] + \text{tr}(x^T_i X_i) - (1+b) \sum_{t=t,\Delta}^{N} (c^T_{\Delta x} y_\theta) k_i,
\]
where \( k_t \) is an array of length \( N \) made up of ones for the first array position through to the \( t / \Delta t \) array position and zeros otherwise. Introducing the function \( g(p_t) \) to represent the company’s liquidation liability for production costs \( (c \hat{E}_{t-\Delta t}[q_t | p_t]) \) and for debt \( \left(1 + b \right) \sum_{i=t/\Delta t}^{N} \left( e_i^T y_{i\eta} \right) k_t \), less both the net hedge payoff of maturing contracts and the market value of live hedge contracts \( (\text{tr}(x_0^T X_t)) \), then the liquidation cash-flow term can be rewritten:

\[
L_t = p_t q_t - \left[ c \hat{E}_{t-\Delta t}[q_t | p_t] \right] + \left(1 + b \right) \sum_{i=t/\Delta t}^{N} \left( e_i^T y_{i\eta} \right) k_t - \text{tr}(x_0^T X_t)
\]

\[
= p_t q_t - g(p_t).
\]

The distribution of \( L_t \) to non-equity stakeholders is capped at the total face-value of outstanding debt. Any remainder is paid to equity as a taxable liquidating dividend (\( GL_t \)):

\[
GL_t = \begin{cases} 
(1-\alpha)L_t, & L_t \geq 0 \cap F_t < -E_{t-\gamma}/(1+\gamma) \\
0, & \text{otherwise} 
\end{cases}
\]

\[
= (1-\alpha) \left\{ p_t q_t - g(p_t) \right\} \left[ 1 - \left[ \frac{E_t - E_{t-\gamma}}{1+\gamma} \right] \right].
\]

Therefore:

\[
\hat{E}_{t-\Delta t}[GL_t | p_t] = (1-\alpha) p_t \left\{ \text{ADP}_{q_t < X_t} - \text{ADP}_{q_t < \text{min}[X_t, Z_t]} + \text{ADP}_{q_t < \text{min}[X_t, Z_t]} \right\}
\]

\[-(1-\alpha)g(p_t) \left\{ \text{CDP}_{q_t < X_t} - \text{CDP}_{q_t < \text{min}[X_t, Z_t]} + \text{CDP}_{q_t < \text{min}[X_t, Z_t]} \right\}
\]

\[-(1-\alpha) p_t \left\{ \text{ADP}_{q_t < \text{min}[g(p_t)/p_t, X_t]} - \text{ADP}_{q_t < \text{min}[g(p_t)/p_t, X_t]} + \text{ADP}_{q_t < \text{min}[g(p_t)/p_t, X_t]} \right\}
\]

\[+ (1-\alpha)g(p_t) \left\{ \text{CDP}_{q_t < \text{min}[g(p_t)/p_t, X_t]} - \text{CDP}_{q_t < \text{min}[g(p_t)/p_t, X_t]} + \text{CDP}_{q_t < \text{min}[g(p_t)/p_t, X_t]} \right\} \].

The total value attributable to equity immediately prior to any dividend payment at time \( t \) (\( E_{t-\gamma} \)) is given by:

\[
E_{t-\gamma} = G_t + GL_t + E_{t-\gamma} \left[ 1 - \left[ \frac{E_t - E_{t-\gamma}}{1+\gamma} \right] \right].
\]

Therefore:
\[
\hat{\mathbb{E}}_{t-t^*}[E_{t^*} \mid p_t] = \hat{\mathbb{E}}_{t-t^*}[G_t \mid p_t] + \hat{\mathbb{E}}_{t-t^*}[GL_t \mid p_t]
\]
\[+ (E_{t^*} \mid p_t) \left\{ 1 - \text{CDP}_{q_t < Z_t} + \text{CDP}_{q_t < \min[Z_t, \chi_t]} - \text{CDP}_{q_t < \min[Z_t, \chi_t]} \right\}. \tag{A.1}
\]

**Cash-flows to debt and against the limited liability option**

The model company’s non-equity stakeholders are the debt finance providers, the hedge contract providers, the providers of production labour and equipment, and the sources of the bankruptcy deadweight costs; together they also provide equity with its limited liability option. The periodic combined cash-flow to debt and against the limited liability option \((H_t)\) equals: the total due face-values of maturing bonds, under condition of solvency or financial distress (i.e. non-bankruptcy); or, in the event of bankruptcy (which entails termination of all outstanding debt), the total face-value of all outstanding debt less any liquidation cash-flow shortfall. That is, given bankruptcy, a positive liquidation cash-flow means that all non-equity claims can be satisfied in full, but a negative liquidation cash-flow represents a claim by equity against its limited liability option. Therefore:

\[
H_t = \begin{cases} 
\left( \mathbf{e}_t^T \mathbf{y}_t \right) \mathbf{1} , & F_t \geq -E_{t^*} / (1 + \gamma) \\
\sum_{i=t^*}^N \left( \mathbf{e}_{t^*}^T \mathbf{y}_t \right) \mathbf{k}_i , & L_t \geq 0 \cap F_t < -E_{t^*} / (1 + \gamma) \\
L_t + \sum_{i=t^*}^N \left( \mathbf{e}_{t^*}^T \mathbf{y}_t \right) \mathbf{k}_i , & L_t < 0 \cap F_t < -E_{t^*} / (1 + \gamma)
\end{cases}
\]

\[
= \left( \mathbf{e}_t^T \mathbf{y}_t \right) \mathbf{1} + \left\{ \sum_{i=t^*}^N \left( \mathbf{e}_{t^*}^T \mathbf{y}_t \right) \mathbf{k}_i - \left( \mathbf{e}_t^T \mathbf{y}_t \right) \right\} \mathbf{1} \frac{I}{F_t - E_{t^*} / (1 + \gamma)} + L_t \mathbf{1} \underbrace{\text{CDP}_{q_t < \min[Z_t, \chi_t]} + \text{CDP}_{q_t < \min[Z_t, \chi_t]}}_{\text{CDP}}.
\]

Therefore:

\[
\hat{\mathbb{E}}_{t-t^*}[H_t \mid p_t] = \left( \mathbf{e}_t^T \mathbf{y}_t \right) \mathbf{1} + \left\{ \sum_{i=t^*}^N \left( \mathbf{e}_{t^*}^T \mathbf{y}_t \right) \mathbf{k}_i - \left( \mathbf{e}_t^T \mathbf{y}_t \right) \right\} \left\{ \text{CDP}_{q_t < \min[Z_t, \chi_t]} - \text{CDP}_{q_t < \min[Z_t, \chi_t]} + \text{CDP}_{q_t < \min[Z_t, \chi_t]} \right\}
\]

\[
+ p_t \left\{ \text{ADP}_{q_t < \min \left[ \frac{g(p_t)}{p_t}, Z_t \right]} - \text{ADP}_{q_t < \min \left[ \frac{g(p_t)}{p_t}, Z_t, \chi_t \right]} + \text{ADP}_{q_t < \min \left[ \frac{g(p_t)}{p_t}, Z_t, \chi_t \right]} \right\}
\]

\[
- g(p_t) \left\{ \text{CDP}_{q_t < \min \left[ \frac{g(p_t)}{p_t}, Z_t \right]} - \text{CDP}_{q_t < \min \left[ \frac{g(p_t)}{p_t}, Z_t, \chi_t \right]} + \text{CDP}_{q_t < \min \left[ \frac{g(p_t)}{p_t}, Z_t, \chi_t \right]} \right\}.
\]
Consider that risky debt is a combination of risk-free debt plus a short position in a limited liability option that has some individualised exercise priority amongst all the various limited liability options provided by non-equity stakeholders to equity. This model is not concerned with separating out the specific limited liability exposure of debt from other non-equity stakes. Instead the value of equity’s comprehensive limited liability option underwritten by all of the non-equity stakes is of interest. To obtain this value it is necessary to separate out the risk-free debt valuation from the total value of debt and the short limited liability option. Thus define $D_{t-}$, $D_t$ and $D_{t+}$ to each be the risk-free value of all outstanding bonds at time $t$ but respectively at the immediate instances: before settlement of maturing bonds and before issue of any new bonds; after settlement of maturing bonds and before issue of any new bonds; and after settlement of maturing bonds and after issue of any new bonds. Therefore:

$$D_{t-} = y_{0,t} + y_{+t,\Delta t} + \cdots + y_{t_-,\Delta t} + \frac{y_{0,\Delta t} + y_{+\Delta t,\Delta t} + \cdots + y_{t_-,\Delta t + \Delta t}}{e^{\tau t}}$$

$$y_{0,t} + y_{+t,\Delta t} + \cdots + y_{t_-,\Delta t} + \frac{y_{0,\Delta t} + y_{+\Delta t,\Delta t} + \cdots + y_{t_-,\Delta t + \Delta t}}{e^{\tau t}} = \sum_{i=(t-\Delta t)/\Delta t}^{N} \left( \frac{\mathbf{e}^T \mathbf{y}_i}{(i-\Delta t)r\Delta t} \right) \mathbf{k}_i$$

$$D_t = \frac{y_{0,t} + y_{+t,\Delta t} + \cdots + y_{t_-,\Delta t} + \Delta t}{e^{\tau t}}$$

$$y_{0,t} + y_{+t,\Delta t} + \cdots + y_{t_-,\Delta t} + \frac{y_{0,\Delta t} + y_{+\Delta t,\Delta t} + \cdots + y_{t_-,\Delta t + \Delta t}}{e^{\tau t}} = \sum_{i=(t-\Delta t)/\Delta t}^{N} \left( \frac{\mathbf{e}^T \mathbf{y}_i}{(i-\Delta t)r\Delta t} \right) \mathbf{k}_i$$

$$D_{t+} = y_{0,t} + y_{+t,\Delta t} + \cdots + y_{t_-,\Delta t} + y_{t_+,\Delta t} + \frac{y_{0,\Delta t} + y_{+\Delta t,\Delta t} + \cdots + y_{t_-,\Delta t + \Delta t}}{e^{\tau t}}$$

$$y_{0,t} + y_{+t,\Delta t} + \cdots + y_{t_-,\Delta t} + y_{t_+,\Delta t} + \frac{y_{0,\Delta t} + y_{+\Delta t,\Delta t} + \cdots + y_{t_-,\Delta t + \Delta t}}{e^{\tau t}} = \sum_{i=(t+\Delta t)/\Delta t}^{N} \left( \frac{\mathbf{e}^T \mathbf{y}_i}{(i+r)\Delta t} \right) \mathbf{k}_i$$

It is assumed that equity must ‘purchase’ a new comprehensive limited liability option each period: $O_{t-} \leq 0$ symbolises the short payoff value of the limited liability option expiring at time $t$; and $O_{t+} \leq 0$ symbolises the short initial value of the limited liability option newly contracted at time $t$ for the ensuing period. Equity’s periodic expense for the
limited liability option is incorporated in the model as an offset against the risk-free valuation of new debt proceeds. A personal tax penalty for debt income relative to equity income is also incorporated into the risk-free valuation of new debt proceeds. Hence the risky measure of new debt proceeds \((Y_t)\) is:

\[
Y_t = O_t + \frac{y_{i+2N}}{e^{(\alpha - A_a)}T} + \frac{y_{i+2N+2}}{e^{(\alpha - A_a)}T^2} + \ldots + \frac{y_{i+N}}{e^{(\alpha - A_a)}T^{N-1}}
\]

where \((\alpha - A_a)\) is the personal tax penalty acting against the corporate tax shield benefit of debt (i.e. \(\alpha\) is the corporate tax rate and \(A_a\) is the marginal investor’s effective rate of combined personal and corporate tax being shielded by debt finance).

Normally the cost of the limited liability option associated with risky zero-coupon bonds is deducted from the upfront financing proceeds. For this model set-up the cost of the debt-specific limited liability option is paid in variable instalments at the start of each period of the debt’s life (which for the case of single-period bonds is equivalent to a single upfront deduction from financing proceeds). Additionally the company must pay upfront each period for the limited liability exposure of other non-equity stakeholders. This payment will vary from period to period and can be conceptualised as a ‘sign-on’ expense for labour, equipment and service suppliers plus a risk-premium transaction cost for hedge contracts.\(^{14}\)

The cost of the comprehensive limited liability option for any period may be such that the risky new debt proceeds measure is zero or negative; this will naturally be the case if no new debt is actually being issued \((1^T(\eta_{+e} e_{+s}) = 0)\). Nevertheless the effect is that the comprehensive limited liability option is incorporated into the model company as a periodic and variable tax deductible expense.

The total value attributable to debt and non-equity’s expiring short limited liability option at time \(t\) immediately prior to any payment to non-equity and before any issue of new debt or contracting of a new limited liability option \((D_{-} + O_{-})\) is given by:

\(^{14}\) The hedge contract providers are assumed to not pose any default risk to the company.
\[ D_{t-} + O_{t-} = H_t + D_t \left\{ 1 - \frac{1}{F_{t, <E_{t-} >}} \right\}. \]

Therefore:
\[ \hat{E}_{t-\Delta t} [D_{t-} + O_{t-} | p_t] = \hat{E}_{t-\Delta t} [H_t | p_t] + (D_t | p_t) \left\{ 1 - CDP_{q_t < Z_t} + CDP_{q_t < \min \{Z_t, Z_t\}} - CDP_{q_t < \min \{Z_t, Z_t\}} \right\}. \] (A.2)

Valuation method

Assume that at time \( t = N\Delta t \) the company either ceases operations or becomes an all-equity company. Then the ongoing debt value and short comprehensive limited liability option value are zero and no new debt is issued \( (D_{N\Delta t^+} = O_{N\Delta t^+} = Y_{N\Delta t} = 1^r (y, e_{N\Delta t^+}) = 0) \).

The ongoing equity value \( (E_{N\Delta t^+}) \) is zero if the company ceases operations or otherwise can be set equal to some estimate (e.g. from a basic present value analysis). It is then possible to solve for \( \hat{E}_{(N-1)\Delta t} [E_{N\Delta t^-} | p_{N\Delta t^-}] \) and \( \hat{E}_{(N-1)\Delta t} [D_{N\Delta t^-} + O_{N\Delta t^-} | p_{N\Delta t^-}] \) for all possible \( p_{N\Delta t^-} \). Solutions obtain for equity and for debt plus short comprehensive limited liability option values at each price-node by continuing backwards through the price-tree with risk-neutral valuation such that:
\[ E_{(t-\Delta t)^+} = e^{-r\Delta t} \hat{E}_{t-\Delta t} [E_{t-}], \] (A.3)
\[ D_{(t-\Delta t)^+} + O_{(t-\Delta t)^+} = e^{-r\Delta t} \hat{E}_{t-\Delta t} [D_{t-} + O_{t-}]. \] (A.4)

APPENDIX B. LEVERAGE, HEDGING AND FINANCIAL RISK MEASURES

The chosen measure of leverage \( (\ell_{t^+}) \) for the model company is the risk-free valuation of newly issued bonds \( (D_{t^+}) \) divided by ongoing equity value \( (E_{t^+}) \):
\[ \ell_{t^+} = D_{t^+} / E_{t^+}. \]
The chosen measure of hedging is based on Tufano’s (1996) delta-percentage technique. The value at time \( t \) of an individual hedge position entered at time \( \tau \leq t \) and maturing at time \((\tau + \kappa \Delta t) > t\) is \( x_{\tau, \tau + \kappa \Delta t} X_{\tau, \tau + \kappa \Delta t} \) (see Appendix A for the formulation). Assuming geometric Brownian motion for the underlying price process, the delta of the individual hedge position (i.e. the sensitivity of the value of the hedge position with respect to the underlying price) is:

\[
\frac{\partial}{\partial p_t} x_{\tau, \tau + \kappa \Delta t} X_{\tau, \tau + \kappa \Delta t} = -e^{-\delta (\tau + \kappa \Delta t - t)} x_{\tau, \tau + \kappa \Delta t} \left[ w_{\tau, \tau + \kappa \Delta t} + (1 - w_{\tau, \tau + \kappa \Delta t}) \Phi \left\{ \frac{-\ln \left( \frac{p_t}{z_{\tau, \tau + \kappa \Delta t}} \right) - \left( r - \delta + \frac{\sigma_p^2}{2} \right) (\tau + \kappa \Delta t - t)}{\sigma_p \sqrt{\tau + \kappa \Delta t - t}} \right\} \right]
\]

where: \( p_t \) is the unit price of the underlying production output being hedged; \( \sigma_p^2 \) is the annual variance rate for the output price; \( r \) is the annual, continuously compounding risk-free interest rate; \( \delta \) is the annual, continuously compounding convenience yield of the production output; \( x_{\tau, \tau + \kappa \Delta t} \) is the units of hedge quantity; \( 0 \leq w_{\tau, \tau + \kappa \Delta t} \leq 1 \) is the ratio choice for hedge quantity committed to short forwards as opposed to put options; \( z_{\tau, \tau + \kappa \Delta t} \) is the strike price of the put options; and \( \Phi \{ \} \) is the standard normal cumulative distribution function.

Given that a binomial price process is assumed for model implementation, the corresponding discrete delta measure for the individual hedge position is:

\[
\frac{\Delta}{\Delta p_t} x_{\tau, \tau + \Delta T} X_{\tau, \tau + \Delta T} = -e^{-\delta (T - 1)} x_{\tau, \tau + \Delta T} w_{\tau, \tau + \Delta T} \left[ \sum_{j=0}^{(T/\tau - 1)} \max \left\{ 0, \frac{z_{\tau, \tau + \Delta T}}{p_t e^{(\tau - \delta - \sigma_p^2/2)\Delta \tau}} e^{(2j-(T-1)/\Delta \tau)\sigma_p \sqrt{\Delta \tau}} \left( e^{\sigma_p \sqrt{\Delta \tau}} - e^{-\sigma_p \sqrt{\Delta \tau}} \right) \right\} \right] m'(1 - m)^{(T/\tau - 1 - j)}
\]

\[
+ x_{\tau, \tau + \Delta T} (1 - w_{\tau, \tau + \Delta T}) \left[ \sum_{j=0}^{(T/\tau - 1)} \max \left\{ 0, \frac{z_{\tau, \tau + \Delta T}}{p_t e^{(\tau - \delta - \sigma_p^2/2)\Delta \tau}} e^{(2j-(T-1)/\Delta \tau)\sigma_p \sqrt{\Delta \tau}} \left( e^{\sigma_p \sqrt{\Delta \tau}} - e^{-\sigma_p \sqrt{\Delta \tau}} \right) \right\} \right] m'(1 - m)^{(T/\tau - 1 - j)}
\]

\[
- x_{\tau, \tau + \Delta T} (1 - w_{\tau, \tau + \Delta T}) \left[ \sum_{j=0}^{(T/\tau - 1)} \max \left\{ 0, \frac{z_{\tau, \tau + \Delta T}}{p_t e^{(\tau - \delta - \sigma_p^2/2)\Delta \tau}} e^{(2j-(T-1)/\Delta \tau)\sigma_p \sqrt{\Delta \tau}} \left( e^{\sigma_p \sqrt{\Delta \tau}} - e^{-\sigma_p \sqrt{\Delta \tau}} \right) \right\} \right] m'(1 - m)^{(T/\tau - 1 - j)}
\]
where: 

\[ T = \left( \tau + \kappa \Delta t - t \right) \]

is the time remaining until hedge contract maturity such that 
\( \tau, t, \tau + \kappa \Delta t \in \{0, \Delta t, 2 \Delta t, ..., N \Delta t\} \), \( \tau \leq t < \tau + \kappa \Delta t \), and positive integer \( N \) is the number of 
controlled production periods; \( t = \Delta t / n \) is the discrete time-step of the binomial process 
being as that a single production period has duration \( \Delta t \) years and \( n \) binomial time-steps; 
and 
\[ m = \left( e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}} \right) / \left( e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}} \right). \]

The hedging measure \( (h_{t+}) \) is named the hedge-delta ratio and is defined to be the negative 
of, the ratio of, the delta of the total portfolio of new and live hedge positions, to the total 
expected quantity of future production output:

\[
h_{t+} = \frac{\sum_{j=1}^{N} \Delta X_{0,j\Delta t}X_{t,j\Delta t} - \sum_{j=0}^{\left\lfloor t/\Delta t \right\rfloor + 1} \Delta P_{j} \sum_{j=0}^{\left\lfloor t/\Delta t \right\rfloor + 1} \Delta X_{i\Delta t,j\Delta t}X_{i\Delta t,j\Delta t}}{R - \sum_{j=0}^{\left\lfloor t/\Delta t \right\rfloor} \hat{E}_{(j-1)\Delta t} [q_j \mid P_{j\Delta t}]} \]

for \( t \in \{\Delta t, 2 \Delta t, ..., (N-1)\Delta t\} \), and where \( R \) is the initial total expected quantity of future 
production output.

Four measures of financial risk are defined: the value of equity’s comprehensive limited 
liability option relative to ongoing equity value \( (-O_{t+} / E_{t+}) \); two probability of bankruptcy 
\( (\Pr_{t+}, \hat{\Pr}_{t+}) \) measures; and an output price beta \( (\hat{\beta}_{t+}) \) measure. \( \Pr_{t+} \) is the probability 
of bankruptcy over the preceding production period due to production quantity risk:

\[
\Pr_{t+} = \hat{\Pr}_{t+} \left[ \frac{1}{(1+\gamma)} \right] 
\]

where: the indicator function, \( I_{\text{logical statement}} \), equals one if the \text{logical statement} is true and zero otherwise; and \( F_t < -E_{t+} / (1+\gamma) \) is the free-cash-flow condition for instigation of bankruptcy.

\( \hat{\Pr}_{t+} \) is the risk-neutral probability of bankruptcy for the ensuing production period:

\[
\hat{\Pr}_{t+} = \hat{E}_{t+} \left[ \frac{1}{(1+\gamma)} \right].
\]
\( \hat{\beta}_{t^+} \) is the risk-neutral sensitivity of equity’s rate of return to the underlying output price rate of return for the ensuing production period (i.e. equity’s output price beta):

\[
\hat{\beta}_{t^+} = \frac{\hat{E}_t \left( \frac{p_{t+M}}{p_t e^{-\delta M}} - 1 \right) \left( \hat{E}_t \left[ \frac{E_{t(\tau,\Delta^L)} - p_{t+M}}{E_{t}} - 1 \right] \right)}{\hat{E}_t \left( \frac{p_{t+M}}{p_t e^{-\delta M}} - 1 \right)^2 - \hat{E}_t \left[ \frac{p_{t+M}}{p_t e^{-\delta M}} - 1 \right]^2}.
\]

REFERENCES


