Forward Guidance at the Zero Lower Bound

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Abstract

I develop a fast, efficient and accurate way to approximate the zero lower bound (ZLB) that can solve complicated state-of-the-art DSGE models. This method opens up the study of previously infeasible problems related to the effect of the ZLB on macroeconomic outcomes. The method is based on the idea that a ZLB constraint that is expected to last for multiple periods can be conceptualized as a predictable sequence of contractionary monetary policy shocks. Using this insight, the method iterates on agents’ expectations of the policy interest rate until those expectations respect the ZLB and, eventually, a return to Taylor-rule policy. The appeal is that the exit from the ZLB is determined endogenously. It is capable of solving models with a large number of state variables such as a Smets-Wouters model, and can accommodate announcements about the path of future policy. In this paper, I study one such type of announcement, forward guidance since the ZLB became binding in the US in 2009, and argue that it has been extremely effective. Using an estimated Smets-Wouters model, I find that in the absence of stabilizing actions by the Federal Reserve, the aggregate shocks to the economy since 2009 would have generated much larger output losses and greater variability in output and inflation than has been observed. The results suggest that, in response to shocks, workers and firms have set the window of expected ZLB duration to at least two years each quarter since 2009. That is, the expected ZLB duration period has been a rolling window of at least two years.

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1 Introduction

The zero lower bound has now been binding on the Federal Reserve’s policy interest rate, the Federal Funds rate, for over five years. Documenting the consequences of the zero lower bound in New Keynesian dynamic stochastic general equilibrium (DSGE) models has attracted much attention over this period (see, for example, [Wieland 2014]). In addition, understanding how monetary policy has operated, and how it should operate, when faced with the constraint of the zero lower bound has been of particular urgency given the relatively weak performance of the US economy since the financial crisis in late 2008. This paper contributes to this understanding by studying the role that time-dependent forward guidance plays in smoothing shocks which impact the economy at the zero lower bound.

A study of the zero lower bound in DSGE models that resemble those used in policy analysis requires an efficient and extremely fast way to compute the equilibrium path of the economy, and thereby accommodate how those in policy positions use such models. To this end, I develop an intuitive implementation of the zero lower bound which builds on existing linear techniques. The method opens up a new set of previously infeasible research questions. If we think of the zero lower bound as a predictable restriction on the parameters of the Taylor rule, then the zero lower bound is a change in the structure of the economy which is anticipated by workers and firms. By iterating on the path of the interest rate until it conforms with agents’ expectations, it is straightforward to solve for the state variables of the model subject to this the anticipated structural change. The approach extends [Guerrieri and Iacoviello (2014)] by explicitly allowing for the central bank to communicate and commit to a path of interest rates. It is an approximation, but like [Guerrieri and Iacoviello (2014)], it is an accurate one which delivers substantial computational gains.

I use the implementation to study the consequences of unanticipated random shocks which impact the economy at the zero lower bound, and to ask: to what extent does the Federal Reserve use forward guidance to mitigate shocks when it is constrained by the zero lower bound? This question is motivated by the empirical observation that the variances of inflation and output growth during the period that the zero lower bound has been binding are comparable to the pre-zero lower bound period. In contrast to this observation, as I show in this paper, standard New Keynesian models predict large fluctuations in inflation and output growth when the central bank cannot react to unanticipated shocks. This suggests that the Federal Reserve’s policies are misspecified in New Keynesian models at the zero lower bound and, in particular, the Federal Reserve may be using alternative policy tools to act against unanticipated shocks. I study forward guidance policies, implemented as an announced path of the nominal interest rate. Purchases of long-term bonds with the goal of
pushing down yields on those bonds can be approximated as forward guidance on the short rate, as there is a close relationship between the short-rate and long-term rates through the expectations channel.

If it is true that the Federal Reserve’s active use of forward guidance policy mitigates the effect of the zero lower bound, then the argument that the fiscal multiplier is large at the zero lower bound loses some bite. A number of papers have shown how exogenous increases in spending at the zero lower bound have large effects on the real interest rate by stimulating (directly) inflation. I show that this is not necessarily true in the absence of stabilizing behaviour by the central bank. If the central bank is acting against large inflation deviations through forward guidance, and government spending is stimulating inflation, then the fiscal and monetary authorities could be acting against each other. This is an important point and suggests that perhaps the zero lower bound is not something that fiscal policy should try to counter if it is the case that unconventional monetary policy can effectively smooth out inflation and output growth variances.

There is a growing literature on efficiently accounting for the zero lower bound into DSGE models (for an example application, see Del Negro et al. [2014]). The method of this paper adds to this literature and links it to methodological papers which outline techniques to solve models subject to anticipated structural changes. Extending Guerrieri and Iacoviello [2014], the implementation in this paper explicitly allows for a future anticipated structure of the economy, in addition to accounting for the zero lower bound. The method relies on the insight that the zero lower bound can be implemented by piecewise-combining multiple regimes of the same model. As described by Guerrieri and Iacoviello [2014], under such implementations, two regimes are specified for when the zero lower bound binds and when it does not. The combination of the different regimes, and expectations for the length of time that regime holds generates strong non-linearities in model variables.

I first illustrate the operation of the algorithm using the three-equation New Keynesian model of Ireland [2004]. Using that model, I illustrate some of the main results on unanticipated shocks at the zero lower bound and study whether time-dependent forward guidance, where the length of the period of zero interest rate policy is tied to inflation, improves the overall volatility of inflation and output growth. This question is interesting and relevant because of a trade-off that arises when forward guidance stabilizes output today but exposes the economy to excess volatility from unanticipated shocks in the future. I show through simulations that the benefits from forward guidance depend on the persistence of shocks and the weight the central bank places on output and inflation deviations.

I compute impulse responses following a shock which drives the nominal interest rate to zero when active forward guidance is used, and show that the determinstic paths of inflation
and output reflect the theoretical results of Eggertsson and Woodford (2003) and Werning (2012); that the central bank should hold the nominal rate lower for longer to stabilize output and inflation today and commit to higher inflation and output later. Those optimal time-dependent forward guidance policies are derived in tractable models with few state variables and in the absence of subsequent shocks. The implementation here allows me to study forward guidance in a model with many state variables and where shocks impact the economy each period. I show in simulations of the Ireland (2004) model that these components matter.

Motivated by the results using the Ireland (2004) model, I submit an estimated model of Smets and Wouters (2007) to the zero lower bound algorithm of this paper to study how severe the Great Recession could have been in the absence of stabilizing policy by the Federal Reserve. The Smets and Wouters (2007) model is estimated using the shadow interest rate series developed by Wu and Xia (2014) to obtain the sequence of structural shocks over the period the federal funds rate has been subject to the zero lower bound (up to 2014Q2). I then apply those ‘true’ structural shocks to the model subject to the zero lower bound algorithm to back out the expected length of periods of forward guidance. In this exercise, I exploit the difference between observed variables and counterfactual variables to understand the contribution of forward guidance to the observed series.

Interestingly, the model implies forward guidance was indeed effective in stabilizing the output gap, and, in particular, that the expected length of time that interest rates were expected to be zero started at eight quarters in 2009Q1 and has not declined in subsequent periods. That is, the counterfactual paths imply that agents in the model have anticipated that the zero lower bound will bind for two years in 2009Q1, but that in each subsequent quarter, have expected the policy rate to remain at zero for an additional two to three years. The period of time for which the zero lower bound has been expected to bind has essentially reflected an expanding rolling window in the years following the conflagration of the financial crisis. Policymakers and commentators have discussed there being a ‘new normal’ in US monetary policy. These results suggest that, at least over the period since the Great Recession, the ‘new normal’ is that workers and firms expect the nominal interest rate to remain at zero across the forecast horizon.

The paper proceeds as follows. Section 2 briefly introduces a method of solving linear rational expectations models with foreseen structural changes to ground notation and thinking. Section 3 develops the algorithm and discusses how implementing the zero lower bound can be achieved using this method of solving the model. Section 4 illustrates how the algorithm works using a standard small New Keynesian model, comparing the solution to a full non-linear solution and to recently developed alternative linear methods. Section 4 also
discusses how shocks which impact the economy at the zero lower bound can cause severe fluctuations in output growth and inflation and how forward guidance can mitigate the effect of those shocks. Finally, Section 5 uses evidence from the period of the Great Recession in the United States to draw some conclusions about the severity of the zero lower bound constraint and the role of forward guidance in a fully estimated New Keynesian model which is similar to those used in policy analysis.

2 Solving linear rational expectations models

This section outlines a solution method for solving linear rational expectations models to fix notation for understanding the algorithm used to implement the zero lower bound.

2.1 Linear rational expectations model

Write a linear rational expectations model in matrix form as:

$$\tilde{\Gamma}_0 y_t = \tilde{\Gamma}_1 y_{t-1} + \tilde{C} + \tilde{\Psi} \epsilon_t,$$  \hspace{1cm} (1)

where $y_t$ is the state vector, defined by:

$$y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ E_t z_{t+1} \end{pmatrix},$$

and where $y_{1,t}$ is a $(n_1 \times 1)$ vector of exogenous and endogenous variables, and $y_{2,t}$ is a $(n_2 \times 1)$ vector with those endogenous variables for which conditional expectations appear. The vector $z_{t+1}$ of size $(k \times 1)$ contains leads of $y_{2,t}$. The dimension of $y_t$ is $(n \times 1)$, where $n = n_1 + n_2 + k$. Assume $\varepsilon_t$ to be a $(l \times 1)$ vector of serially uncorrelated processes. The matrices $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ are $(n_1 + n_2 \times n)$ matrices, $\tilde{C}$ is $(n_1 + n_2) \times 1$ and $\tilde{\Psi}$ is $(n_1 + n_2) \times l$.

We seek a solution of the model (1) in the form of a VAR(1). There are $(n_1 + n_2)$ equations in model (1), and so because of the presence of expectations, it is not possible to invert $\tilde{\Gamma}$.

Sims (2002) proposal is to append to the model $k$ expectations revisions $\eta_t = E_t z_t - E_{t-1} z_t$ with $E_t \eta_{t+j} = 0$ for $j \geq 1$ which will be solved as part of the solution. As detailed in the appendix, using these expectations revisions, a solution can be written in the form:

$$y_t = S_0 + S_1 y_{t-1} + S_2 \varepsilon_t,$$

which is the desired VAR(1) specification.
2.2 Foreseen structural changes

Now suppose the structural parameters of the economy are known to change into the future. In particular, suppose the economy is expected to evolve with the following structure: in time period \( t = 1 \), the economy starts with the following structure:

\[
\tilde{\Gamma}_{0,1} y_t = \tilde{\Gamma}_{1,1} y_0 + \tilde{C}_1 + \tilde{\Psi}_1 \varepsilon_1,
\]

and in time periods \( 2 \leq t \leq T \), the structural parameters of the economy evolve according to:

\[
\Gamma_{0,t} y_t = \Gamma_{1,t} y_{t-1} + C_t + \Pi \eta_t + \Psi_t (\varepsilon^u_t + \varepsilon^a_t),
\]

where \( \varepsilon^u_t \) are shocks which are unanticipated at time period \( t = 1 \), \( \varepsilon^a_t \) are shocks which are anticipated at \( t = 1 \) and \( \Pi \) is a \((n \times k)\) matrix specifying the position of expectation revisions. Unanticipated shocks are added to show that it is possible to solve the model subject to foreseen structural changes and unanticipated shocks, though the solution would need to be computed each time period to solve for unanticipated shocks. Notice that that matrices specifying the structural parameters are time-varying. After time period \( T + 1 \), the structural parameters of the economy are fixed, so that the system becomes:

\[
\bar{\Gamma}_0 y_t = \bar{\Gamma}_1 y_{t-1} + \bar{C} + \bar{\Pi} \eta_t + \bar{\Psi} \varepsilon_t.
\]

As discussed in the appendix, Cagliarini and Kulish (2013) show how to solve for the state variable \( y_t \) for all time periods. The necessary condition is that the final (bar) structure of the economy has a solution, which ensures the economy will end up on its saddle path. So long as the economy reaches that saddle path, the intermediate structure can be (almost) unconstrained. Given a unique solution to the final (bar) structure, Cagliarini and Kulish (2013) show that the intermediate path of the economy is unique.

3 Implementing the zero lower bound

Section 2.2 outlined how to solve for the path of an economy with a sequence of anticipated changes to the structural parameters. Here, I argue that the zero lower bound is simply an anticipated change in the structural parameters of the central bank’s policy rule. Let the gross nominal interest rate \( r_t \), expressed as a log deviation from steady-state \( r_{ss} \), be \( \hat{r}_t \). Let a central bank follow a Taylor rule in setting the nominal interest rate, responding to
log deviations in inflation from target $\hat{\pi}_t$, log deviations in the growth rate of output from steady-state $\hat{g}_t$, and log deviations in the output gap from zero $\hat{x}_t$. The central bank displays some inertia in setting the interest rate and is subject to a policy shock $\varepsilon_{r,t}$. The zero lower bound on $r_t$ is:

$$\hat{r}_t = \max\{-r_{ss}, \rho_r\hat{r}_{t-1} + \rho_\pi\hat{\pi}_t + \rho_g\hat{g}_t + \rho_x\hat{x}_t + \varepsilon_{r,t}\}.$$ 

The zero lower bound will only bind if a sequence of shocks pushes the nominal rate below zero. So suppose we have a sequence of shocks which will push the nominal interest rate below zero. Agents in the economy should behave as they expect the central bank to maintain the interest rate at zero for as long as the shocks keep a ‘shadow’ nominal rate below zero, where the shadow rate $\hat{r}_t^*$ is defined as:

$$\hat{r}_t^* = \rho_r\hat{r}_{t-1} + \rho_\pi\hat{\pi}_t + \rho_g\hat{g}_t + \rho_x\hat{x}_t + \varepsilon_{r,t}.$$ 

If the central bank was to maintain the interest rate at zero, then this is an anticipated change in the rule of the central bank. It is no longer acting under the Taylor rule, but is instead constrained by the zero lower bound.

As is clear from the exposition of Section 2.2, an anticipated change in the future structure of the economy affects all endogenous variables before that change. In turn, changes to the endogenous variables affect the path of the nominal interest rate. To ensure the realized path is consistent with agents’ expectations, I develop an iterative algorithm to update the endogenous variables period-by-period. I detail two forms of the algorithm, one for when shocks are anticipated at the initial time period, and one where they are unanticipated each period.

### 3.1 Algorithm for anticipated shocks

Suppose there is a sequence of anticipated shocks $\{\varepsilon^a_\tau\}_{\tau=t}^{\infty}$ and an initial vector of variables $x_{t-1}$ whose path we want. The algorithm is:

1. Under the sequence of anticipated shocks $\{\varepsilon^a_\tau\}_{\tau=t}^{\infty}$, solve for the path of variables in the linearized economy using the anticipated structural changes solution and the initial vector of variables $x_{t-1}$. This gives a path for the nominal interest rate. Call this constructed path $r^i_t = \{\hat{r}^i_\tau\}_{\tau=t}^{\infty}$.

2. Examine the path $r^i_t$. For the first time period where $r^i_t < 0$, set the central bank to abandon the Taylor rule and set $\hat{r} = -r_{ss}$ so that the nominal rate equals zero.
3. Step 2 changes the anticipated structure of the economy. Under this new structure and given the initial vector of variables \( x_{t-1} \), solve for the path of all variables, including \( \hat{r}_t \). Call this constructed path \( r_{t+1}^i = \{ \hat{r}_{\tau+1}^i \}_{\tau=t}^\infty \).

Iterate on steps 2 and 3 until convergence of \( r_t \).

Convergence of \( r_t \) says that there is no change to the central bank’s behavior. The final sequence of \( r_t \) that results is such that agents rationally expect the zero lower bound to impose under the anticipated sequence of shocks.

3.2 Algorithm for period-by-period unanticipated shocks

To work with an sequence of unanticipated shocks is a more involved and requires the algorithm be run period-by-period because the initial state vector changes each period. Suppose there is a sequence of unanticipated shocks \( \{ \varepsilon^u_{\tau} \}_{\tau=t}^\infty \) and an initial vector of variables \( x_{t-1} \). The algorithm becomes:

For each period \( t \):

1. Under the unanticipated shock \( \varepsilon^u_t \) in time period \( t \) and the initial vector of variables \( x_{t-1} \), solve for the path of the economy. This might involve solving for the path under a sequence of anticipated shocks.

2. Examine the path \( r_t^i \). For the first time period where \( r_t^i < 0 \), set the central bank to abandon the Taylor rule and set \( \hat{r} = -r_{ss} \) so that the nominal rate equals zero.

3. Step 2 changes the anticipated structure of the economy. Under this new structure, calculate using the anticipated structural changes solution and the initial vector of variables \( x_{t-1} \) the path of all variables, including \( \hat{r}_t \). Call this constructed path \( r_{t+1}^i = \{ \hat{r}_{\tau+1}^i \}_{\tau=t}^\infty \).

Iterate on steps 2 and 3 until convergence of \( r_t \).

4. Increment \( t \) by one. The initial vector of variables now becomes \( x_t \), which was solved for in steps 1 to 3. Now draw a new vector of unanticipated shocks \( \varepsilon^u_{t+1} \) and solve steps 1 through 3.

The final sequence of \( r_t \) that results is such that agents rationally expect the zero lower bound to impose under the anticipated sequence of shocks and in the case where an unanticipated shock pushes the nominal interest rate to zero.
### 3.3 The occasionally binding constraint method [TBD]

I compare the algorithm developed here to that of Guerrieri and Iacoviello (2014). That paper uses a piecewise linear approximation which shares the same features of this paper’s approach. In their algorithm, the model is linearized around the non-stochastic steady-state when the occasionally binding constraint (in this case the zero lower bound) holds and when it does not. The algorithm begins with a guess of when the economy is subject to the occasionally binding constraint and when it exits the regime under the constraint. Then, under the assumption that no shocks hit the economy after the first period, together with the guesses of which regime applies in each period, the sequence of vectors of state variables can be obtained. The computed path of state variables is then checked against the guess of regimes until the path agrees with that guess.

The following result connects the algorithm of Guerrieri and Iacoviello (2014) with the structural change based algorithm detailed here. See the appendix for the reasoning.

**Result 1. Conjecture.** The occasionally binding constraint method developed by Guerrieri and Iacoviello (2014) yields the same path for the endogenous variables as the foreseen structural change approach developed here.

This result gives insights into the operation of the foreseen structural change approach and means that the encouraging results on the accuracy of the occasionally binding constraint method carry over to this paper. The approach here provides an explicit interpretation of the central bank’s actions – as an abandonment of the Taylor rule and an announcement about future policy – and is flexible enough to incorporate forward guidance as unconventional monetary policy. As will be seen, this expected forward guidance is crucial in understanding the effect of the zero lower bound and the behavior of the Federal Reserve since 2009.

### 4 An example New Keynesian model

I use the three-equation New Keynesian model of Ireland (2004) to illustrate the accuracy of the linear approximation developed in this paper, and to highlight the economic mechanisms studied in the next section using a full estimated Smets and Wouters (2007) model.

#### 4.1 The model

The Ireland (2004) economy consists of households, intermediate and final-goods producing firms, and a central bank. Households maximize lifetime utility and experience shocks to their demand for consumption goods. Intermediate goods-producing firms are subject to a cost
of adjusting their prices and experience two kinds of shocks: first, shocks to the elasticity of
demand for their intermediate good, which affects their desired markup; and second, shocks
to permanent aggregate productivity. The central bank adjusts nominal interest rates in
response to deviations of inflation, deviations of output growth, and deviations of the natural
rate of output from their steady-state values. For a full exposition of the model, see Ireland
(2004) and his accompanying notes. Table 1 provides a description and calibration of the
parameters of the model.

The full non-linear model consists of an equation relating consumption \( c_t \) to output \( y_t \)
and includes a term for price adjustments costs:

\[
y_t = c_t + \frac{\phi}{2} \left( \frac{\pi_t}{\pi} - 1 \right)^2 y_t,
\]

where \( \pi_t \) is the inflation rate. There is an Euler equation:

\[
\frac{1}{\beta} \frac{1}{r_t} \frac{a_t}{c_t} = E_t \left[ \frac{a_{t+1}}{c_{t+1}} \frac{1}{z_{t+1} \pi_{t+1}} \right],
\]

where \( r_t \) is the gross nominal interest rate, \( a_t \) is a demand shock, and \( z_t \) is a permanent
productivity shock, and an equation derived from intermediate goods producing firms optimal
price adjustment:

\[
E_t \left[ \frac{a_{t+1}}{a_t} \frac{y_{t+1}}{c_{t+1}} \left( \frac{\pi_{t+1}}{\pi} - 1 \right) \frac{\pi_{t+1}}{\pi} \right] = \frac{1}{\beta \phi} \frac{y_t}{c_t} \left[ \theta_t - 1 - \theta_t \left( \frac{a_t}{c_t} \right) \frac{\pi_t}{\pi} + \phi \left( \frac{\pi_t}{\pi} - 1 \right) \frac{\pi_t}{\pi} \right],
\]

where \( \theta_t \) is a shock to intermediate goods producing firms’ desired markup. The central bank
follows a Taylor rule and is constrained by the zero lower bound. The zero lower bound on
the gross nominal interest rate requires \( r_t > 1 \), so the rule becomes:

\[
r_t = \max \left[ 1, r_{t-1} \rho_r \pi_t \pi_t \left( \frac{c_t}{a_t} \right) g_t x_t^{\rho_x} \exp(\epsilon_{r,t}) \right],
\]

where \( g_t \) is the growth rate of output from \( t - 1 \) to \( t \) and \( x_t \) is the efficient level of output:

\[
g_t = \frac{y_t}{y_{t-1}},
\]

\[
x_t = \frac{y_t}{a_t^{1/\eta}}.
\]
Markup and demand shocks follow AR(1) processes:

\[
\ln(\theta_t) = (1 - \rho_{\theta}) \ln(\theta) + \rho_{\theta} \ln(\theta_{t-1}) + \varepsilon_{\theta t}
\]

\[
\ln(a_t) = (1 - \rho_a) \ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{at},
\]

while shocks to technology are permanent:

\[
\ln(z_t) = \ln(z) + \varepsilon_{zt}.
\]

4.2 Non-linear solution

I solve the full non-linear solution with a single demand shock to compare the impulse responses of the non-linear solution with the linearized implementation. That is, I solve the model for shocks to \( a_t \) and with \( \rho_r = 0 \), ignoring shocks to \( \theta_t, z_t \) and the monetary policy rule. This reduces the size of the state space so that the state is two dimensional, in \( y_{t-1} \) and \( a_t \). The projection-based algorithm used is based on Fernández-Villaverde et al. (2012) and yields policy functions for all the endogenous variables. See the Appendix for details.

4.3 Linearized model

The algorithm solves linear models expressed as log deviations from their steady-states. Using Ireland’s notation, the three main equations of the model are the log linearized versions of Equations (2) to (5) which become:

\[
\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} - (\hat{r}_t - \mathbb{E}_t \hat{\pi}_{t+1}) + (1 - \omega)(1 - \rho_a)\hat{a}_t
\]

\[
\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \psi \hat{x}_t - \hat{e}_t
\]

\[
\hat{r}_t = \max\{-r_{ss}, \rho_r \hat{r}_{t-1} + \rho_\pi \hat{\pi}_t + \rho_g \hat{g}_t + \rho_x \hat{x}_t + \varepsilon_{r,t}\},
\]

where \( \omega = \frac{1}{\eta} \), \( \hat{e}_t \) is the deviation of a cost-push shock from its steady-state value equal to \( \hat{e}_t = \frac{1}{\phi} \hat{\theta}_t \), and \( \hat{a}_t \) is the deviation of a preference shock from its steady-state value. Note that the log-linearized version of Equation (2) implies \( \hat{c}_t = \hat{y}_t \). Equation (11) is the IS-curve, Equation (12) is the Phillips curve and Equation (13) is the Taylor Rule subject to the zero lower bound. The following equations define the linearized output gap and the growth rate of output:

\[
\hat{x}_t = \hat{y}_t - \omega \hat{a}_t
\]

\[
\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t,
\]
Table 1: Parameters and calibrated values of Ireland (2004) model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^*$</td>
<td>Target rate of annual inflation (per cent)</td>
<td>2.0</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Cost of price adjustment</td>
<td>200</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>Persistence of the interest rate in the Taylor-rule</td>
<td>0.0</td>
</tr>
<tr>
<td>$\rho_x$</td>
<td>Reaction to deviations of inflation from $\pi^*$</td>
<td>2.0</td>
</tr>
<tr>
<td>$\rho_y$</td>
<td>Reaction to deviation of output from steady state</td>
<td>1.5</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>Reaction to output growth</td>
<td>0.05</td>
</tr>
<tr>
<td>$z$</td>
<td>Steady-state level of TFP</td>
<td>1.0</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Inverse of Frisch elasticity of substitution</td>
<td>0.05</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Slope of Phillips curve</td>
<td>0.1</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>Persistence of demand shock</td>
<td>0.5</td>
</tr>
<tr>
<td>$\rho_e$</td>
<td>Persistence of mark-up shock</td>
<td>0.5</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>Standard deviation of demand shock</td>
<td>0.02</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>Standard deviation of mark-up shock</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>Standard deviation of technology shock</td>
<td>0.0109</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>Standard deviation of policy shock</td>
<td>0.0031</td>
</tr>
</tbody>
</table>

where $\hat{y}_t$ is the deviation of output from steady-state. The following equations define the linearized shock processes:

\[
\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{a,t} \tag{16}
\]
\[
\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{e,t} \tag{17}
\]
\[
\hat{z}_t = \varepsilon_{z,t}. \tag{18}
\]

In this model, by the Euler Equation (3), the steady state nominal interest rate is:

\[
r_{ss} = \pi^* \frac{z}{\beta},
\]

where $z$ is the steady-state growth rate of technology. I solve the linearized model using the method of Sims (2002) described in Section 2.1.

4.4 Results

Using the parameters of Table 1, I calculate impulse responses to a negative five standard deviation shock to $a_t$ under the three different solution methods discussed. The impulse responses for $\hat{r}_t$, $\hat{\pi}_t$, and $\hat{y}_t$ are given in Figure 1. The large negative demand shock drives the nominal interest rate to the zero lower bound for two consecutive periods. Inflation and
output growth fall. Notably, the variance in inflation and output growth is higher for the methods which account for the zero lower bound as compared to the fully linear solution. Comparing the impulse responses from the three different solution methods shows that the approximation is good. The impulse response from the method developer in this paper matches exactly the impulse that is achieved by the OccBin algorithm. This confirms the result above, so that the results on the accuracy of the approximation method developed by Guerrieri and Iacoviello (2014) carry over to this paper’s method.

In Figure 2 I illustrate the iterative steps of the algorithm by subjecting the New Keynesian model to an expected sequence of three consecutive (positive) cost-push shocks. These shocks induce deflation and hit at time period 21. They are expected by agents in the model to be in place in time periods 21, 22 and 23. Following the Taylor rule, the central bank lowers nominal rates. Figure 2a shows the path of the nominal rate if the zero lower bound does not bind. The black line gives the bound. Under the sequence of expected cost-push shocks, the nominal interest rate is expected to fall below the zero lower bound at period 23, after the three shocks impact.

In the second iteration of the algorithm, the first period when the bound binds is identified (Figure 2b). For that period, the central bank announces that it is abandoning the Taylor rule and will instead set $\hat{r}_t = -r_{ss}$. From the point of view of agents in time period 21, this is an anticipated change in the behavior of the central bank. This changes endogenously the entire path of the economy, and so changes the path of the interest rate. In response, the central bank would want to lower the nominal rate faster and earlier. Subsequent iterations of the algorithm (Figures 2c and 2d) ensure that the central bank’s behavior is consistent with the endogenous path of the interest rate. In total it takes 11 iterations for the algorithm to complete. Importantly, it completes in fraction of a second.

[TBD] While the impulse response obtained from the method of this paper matches exactly the impulse response calculated using the OccBin algorithm of Guerrieri and Iacoviello (2014), paths which are calculated when the three-equation New Keynesian model is subject to large shocks give different guesses as to when the zero lower bound binds. Figure 3 provides an example, plotting $\hat{r}_t$ when the economy is subject to a large negative productivity shock in the first period, in addition to random shocks to the other stochastic variables (excluding to the interest rate). In this example, the standard deviation of technology shocks, $\sigma_a$ is set at 0.04 (which is in line with estimate of Ireland (2004)). The two methods disagree
at some points along the simulation, most notably for the third period, when a combination of the shocks should push the interest rate above the zero lower bound (according to the solution when the zero lower bound is not imposed). This causes some noticeable differences in the calculated inflation and output growth series.

[Figure 3 about here.]

4.5 Severity of shocks at the zero lower bound

I use the implementation to analyse the severity of shocks when the nominal interest rate is at the zero lower bound. Figure 1 revealed that, when the central bank cannot stabilize a large negative demand shock pushing the nominal interest rate to its lower bound, the inflation decline is pronounced and output growth is more variable. I investigate this further in Figure 4 by plotting the initial response of inflation following demand ($a_t$) shocks of varying size. The non-stochastic response is the bold black line. The red circles on this line indicate when the zero lower bound binds. Also plotted are fancharts for the initial response when the economy is subject to random shocks to the other stochastic variables (except to the policy rule).

For the region in which the zero lower bound does not bind, the initial response is linear. Beyond large, negative $a_t$ shocks, the zero lower bound binds and the response of inflation and output growth is magnified. In addition, the width of the fanchart widens for both inflation and output growth when the zero lower bound binds. This illustrates how, when the central bank is constrained and unable to act against the shock with the nominal interest rate, the effect of unanticipated stochastic shocks is large. This is intuitive: policy functions for inflation and output growth are steeper at the zero lower bound, and so shocks when the central bank is at that bound move the economy along those steeper functions. Figure 5 plots a normalized measure of the width of the bands around the initial response in Figure 4.

[Figure 4 about here.]

[Figure 5 about here.]

To illustrate the severity of shocks at the zero lower bound over time, Figure 6 shows fancharts for the interest rate, inflation and output growth for 100 simulations of the model which is subject to the zero lower bound, and for when it is not subject to the zero lower bound. The shocks are such that the economy is subject to six consecutive quarters of negative two standard deviation preference shocks, in addition to random shocks (but excluding monetary policy shocks). This pushes the nominal interest rate to its zero lower bound for
an extended period. The black line gives a model path which is closest to the median of the simulations, plotted in the dashed line. The fan charts illustrate how, when the zero lower bound binds, some paths are particularly variable, so that the overall variance of inflation and output growth rise over the simulation period.

[Figure 6 about here.]

To deepen the understanding of why shocks at the zero lower bound could cause more volatility in inflation and output growth, I study the New Keynesian model in the familiar aggregate demand and aggregate supply framework as in Kulish and Jones (2011). To find the aggregate supply schedule, substitute equations (14) and (15) into (12) to get:

$$\pi_t = \psi g_t + \hat{s}_t + (\pi - \psi g),$$  \hspace{0.5cm} (19)

where $\hat{s}_t = \beta E_t \hat{\pi}_{t+1} + \psi \hat{y}_{t-1} - \psi \hat{z}_t - \omega \psi \hat{a}_t - \hat{e}_t$. In the space of output growth and inflation ($g_t, \pi_t$), equation (19) expresses inflation as a linear function of output growth, with slope $\psi$ and intercept $\hat{s}_t + (\pi - \psi g)$. Note that the time-varying component of the intercept $\hat{s}_t$ is zero when the economy is on its balanced growth path. Also note that the slope of the curve depends on the degree of nominal price rigidities. That is, as $\psi \to \infty$ prices become fully flexible which implies a vertical aggregate supply curve. Conversely as the cost of price adjustment rises, $\psi \to 0$, implying that the aggregate supply curve flattens.

To obtain the aggregate demand schedule when the zero lower bound does not bind, substitute equations (13), (14) and (15) into (11) to get:

$$\pi_t = -\left(\frac{1 + \rho_g + \rho_x}{\rho_x} \right) g_t + \hat{d}_t + \left(\pi + \frac{1 + \rho_g + \rho_x}{\rho_x} g\right),$$  \hspace{0.5cm} (20)

where $\hat{d}_t = -\frac{1}{\rho_x} \hat{\tau}_{t-1} + \frac{1}{\rho_x} E_t \hat{x}_{t+1} + \frac{1}{\rho_x} E_t \hat{\pi}_{t+1} - \left(\frac{1 + \rho_g}{\rho_x} \right) \hat{y}_{t-1} + \left(\frac{1 + \rho_g}{\rho_x} \right) \hat{z}_t + \omega \left(\frac{1 + \rho_g}{1 - \rho_w}\right) \hat{a}_t - \frac{1}{\rho_x} \hat{\varepsilon}_{r,t}$. Note that, as for the time-varying intercept in the aggregate supply curve, when the economy is on its balanced growth path, $\hat{d}_t$ is zero. The slope of the curve (20) depends on the parameters of the policy rule. A greater response to deviations of inflation from target, $\rho_x$, flattens the curve. Vice versa, stronger responses to output growth, $\rho_g$, and the output gap, $\rho_x$, steepen the aggregate demand curve. The central bank uses changes in the nominal interest rate to affect growth and stabilize the inflation rate around the target.

The AD and AS curves reveal that shocks move both schedules simultaneously and so, to determine the overall effect of the shocks on $\pi_t$ and $g_t$, we need to find the intersection of these curves. With the values of $\hat{s}_t$ and $\hat{d}_t$ in hand, the aggregate supply curve (19) and the aggregate demand curve (20) can be written as a system of two equations in two variables.
$g_t$ and $\pi_t$ and inverting under reasonable parameter values to get:

$$
\begin{bmatrix}
\pi_t \\
g_t
\end{bmatrix} = \begin{bmatrix}
\pi + \frac{\psi p}{1+\rho_\pi + \rho_\pi} \hat{d}_t + \frac{1+\rho_\pi + \rho_\pi}{1+\rho_\pi + \rho_\pi} \hat{s}_t \\
g + \frac{\rho_x}{1+\rho_\pi + \rho_\pi} \hat{d}_t - \frac{\rho_x}{1+\rho_\pi + \rho_\pi} \hat{s}_t
\end{bmatrix}.
$$

(21)

Now suppose the zero lower bound binds. The AD curve changes, while the AS curve is unchanged. To derive the new AD curve, notice that the rule (13) says that, at the zero lower bound, the parameters of the rule are equal to zero. This says the aggregate demand curve becomes vertical at a level of $g_t$ determined by substituting (13) with the constraint binding, (14) and (15) into (11) to get:

$$
g_t = \tilde{d}_t + g,
$$

where $\tilde{d}_t = r_{ss} + E_t \hat{x}_{t+1} + E_t \hat{n}_{t+1} - \hat{y}_{t-1} + \hat{z}_t + (\omega + (1 - \omega) \rho_a) \hat{a}_t$. The vertical component of the AD curve says that when the zero lower bound binds, the central bank cannot engineer an expansion of output by lowering the nominal interest rate.

The figure below illustrates the AD and AS curves, and motivates why shocks may increase the volatility of inflation and output growth, as a given shock affects inflation and output growth differently depending on whether the equilibrium lies on the sloped or vertical segment of the aggregate demand curve. A given shift in the AS curve along the vertical component of AD cause inflation to be more volatile, while a given shift in the AD curve when the equilibrium lies in the vertical component of the AD curve causes both inflation and output growth to be more volatile.

To determine inflation and output growth at an equilibrium where the zero lower bound binds, write the AD and AS equations in inflation and output growth space, and solve to obtain:

$$
\begin{bmatrix}
\pi_t \\
g_t
\end{bmatrix} = \begin{bmatrix}
\pi + \hat{s}_t + \psi \hat{d}_t \\
g + \hat{d}_t
\end{bmatrix}.
$$

(22)

Comparing (21) with (22) reveals that shocks to $\epsilon_t$, $a_t$ and $z_t$ move both aggregate demand and aggregate supply simultaneously, both directly through the shock and indirectly through effects on expected inflation and the output gap. This implies that a full analysis of the effect of shocks on the volatility of inflation and output growth requires solving for the path of all variables including for expected inflation and output growth.
4.6 Time-dependent forward guidance at the zero lower bound

Time-dependent forward guidance – or the explicit announcement of the length of time for which the policy rate is to be held at zero – has become an important tool of central banks around the world in providing monetary stimulus when the zero lower bound binds. By providing credible information about the future path of interest rates, the central bank could encourage households to intertemporally bring forward consumption plans to simulate inflation. The implementation developed in this paper is well suited to analyzing such policies.

To make explicit the idea that forward guidance is stimulatory, as in Eggertsson and Woodford (2003) and Wieland (2014), consider a continuous time New Keynesian model with consumers having household utility:

\[ U(t) = \frac{C(t)^{1-\gamma} - 1}{1 - \gamma} - v(L(t)). \]
The continuous time Euler equation becomes:

\[ \mathbb{E}_t [d \ln C(t)] = \gamma^{-1} [i(t) - \pi(t) - \rho] \, dt, \]

where \( \rho \) is the continuous time discount factor. Solving the Euler equation forward and taking limits gives:

\[ \ln C(t) = -\gamma^{-1} \mathbb{E}_t \int_0^\infty [i(t + s) - \pi(t + s) - \rho] \, ds + \mathbb{E}_t \lim_{\tau \to \infty} \ln C(\tau). \]

Suppose the zero lower bound is expected to bind for \( T \) periods. The central bank has control over \( i(t) \) for all \( t \). If by announcing that they will hold \( i(t) \) at zero for a longer period of time than is expected under the sequence of shocks, the central bank causes expected future real interest rates to fall, then through the Euler equation and the expectations channel, that announcement reduces the future value of consumption relative to today’s consumption. Given a supply side where \( \pi(t) \) rises when \( C(t) \) rises, then the central bank’s forward guidance pushes up inflation today. This leads to the next result.

**Result 2.** If, using forward guidance, a central bank constrained by the zero lower bound lowers the path of expected future real interest rates, then output and inflation both rise today.

The following figure illustrates this result in the aggregate demand and aggregate supply framework. Under forward guidance which lowers the future path of nominal interest rates and increases expected inflation and the output gap, both the aggregate demand and aggregate supply curves rise, and current inflation and output growth increase.

Notice that the resulting equilibrium value of \( \pi_t \) and \( \gamma_t \) is closer to what would have resulted if there was no lower bound on the nominal interest rate. The central bank uses announcements about the policy rate to unkink the policy function. This smooths out potentially large fluctuations in inflation and output growth. Also note that under forward guidance, the aggregate demand curve is vertical for all values of inflation. This is because the central bank has abandoned the Taylor rule when making the announcement about future interest rates. This is an interesting observation, indicating that a central bank which commits to holding the interest rate constant for a period exposes the economy to volatility from movements in aggregate supply. The tradeoff between stabilizing output and inflation today against possible future output and inflation volatility tomorrow is examined in the next section.

Is it possible to put some structure on the period of forward guidance, or to link the length of the period of forward guidance to observables? To examine this, I consider a simple time-dependent rule for the announcement policy, where the length of forward guidance of interest
rates at zero is explicitly linked to the rate of inflation that would prevail in the absence of active policy. The central bank, in normal times when the zero lower bound does not bind, follows a standard Taylor rule of the form of Equation (13). When the zero lower bound binds, however, I consider a rule where the central bank forecasts the rate of inflation under the prevailing shocks and forecasts the anticipated length of time the zero lower bound will bind. Knowing these values, it announces a lengthening of the time period for which it expects to hold the nominal interest rate at zero for. Explicitly, the rule takes the form:

\[ T = T^{\text{ant}} + \lceil -\gamma_\pi \hat{\pi}_t^{\text{ant}} \rceil, \]  

(23)

where \( T^{\text{ant}} \) represents the length of time the zero lower bound is expected to bind under the sequence of shocks which push the nominal rate towards zero before forward guidance is provided, and \( \hat{\pi}_t^{\text{ant}} \) represents the inflation rate which would prevail under no forward guidance policy. The value \( \lceil -\gamma_\pi \hat{\pi}_t^{\text{ant}} \rceil \) represents the ceiling of \( -\gamma_\pi \hat{\pi}_t^{\text{ant}} \), so that, for example, \(-2.5\) becomes \(-2\) (this reflects the discrete nature of the announcement period). The parameter \( \gamma_\pi \) is a positive number. A rule such as this says that if inflation is significantly negative, so that \( \hat{\pi}_t^{\text{ant}} \) is a large negative number, then the central bank will keep the interest rate at
zero for longer than is expected under the sequence of shocks causing the zero lower bound to bind. Under Result 2 this is stimulatory at time $t$.

To illustrate the operation of the rule, under the scaling factor of variables in the Ireland (2004) model, I parameterize $\gamma_\pi = 200$. The initial responses of inflation to $\alpha_t$ shocks of different sizes are provided in Figure 7. The length of additional forward guidance is illustrated by the difference between the purple line and the black line of the second panel of the figure. Notice that the active forward guidance policy causes the initial response of inflation and output growth when the zero lower bound binds to roughly replicate the linear policy function. Also note that if there are random shocks to all stochastic variables, the bands around the initial response are tighter. The discreteness of the announcements, however, causes jumps in the initial reaction at larger negative values of the demand shock.

[Figure 7 about here.]

Figure 8 plots impulse responses to a large, single negative demand shock when the central bank follows (23) when the policy rate is constrained by the zero lower bound. The active policy at the zero lower bound with $\gamma_\pi = 200$ calls for the central bank providing forward guidance by announcing that it will hold the nominal interest rate at zero for two periods longer than the zero lower bound is expected to bind. This pushes up inflation contemporaneously and across the entire forecast horizon. By committing to hold interest rates at zero for longer than expected, the central bank accepts some inflation in later periods, which is consistent with the theoretical results suggested by Eggertsson and Woodford (2003) and Werning (2012). Output growth is also subject to a less pronounced decline in the period the shock hits, and subsequently a slower recovery. The volatility of output growth falls.

[Figure 8 about here.]

4.7 Optimal time-dependent forward guidance policy

The previous section established that the deterministic paths of inflation and output growth are less volatile when forward guidance is used when the zero lower bound is expected to bind. The previous sections also showed that, if the economy is subject to shocks while at the zero lower bound, then the volatility of inflation and output growth could increase. There is, then, a trade-off that the central bank could exploit by using a forward guidance rule at the zero lower bound. The possibility that forward guidance can ‘unkink’ policy functions raises a number of interesting questions including: how strong should the central bank tie its announcements to the realized shocks; does the central bank face the same tradeoff between inflation and output when making announcements at the zero lower bound; and, do these
policies cause improvements in overall macroeconomic stability and is this sensitive to the particular model or parameters of the model?

I examine these questions by simulating the Ireland (2004) New Keynesian model for 500 periods, 100 times, under the case where the central bank is actively using forward guidance when the zero lower bound binds and when the central bank is inactive in response to shocks which push the nominal interest rate to zero. I compare the volatility of inflation and of the output gap under the two policies for the same set of shocks. If the central bank uses forward guidance, to keep the analysis focused on rule-based policies, I assume that it commits to completing its announced period of forward guidance. This is an important assumption because it is this commitment which exposes the central bank to excess volatility during the announcement period.

Across the simulation, I compute the following loss function, which can be motivated by a central bank which maximizes households’ utility function over time:

\[
\text{Loss} = \text{var}(\hat{\pi}_t) + \lambda \text{var}(\hat{x}_t).
\]

The value of \( \lambda \) determines how the central bank weights the variance of inflation relative to the variance of the output gap.

Figures 9 and 10 plot the median values of \( \text{var}(\hat{\pi}_t) \), \( \text{var}(\hat{x}_t) \) and the loss when \( \lambda = 0.2 \), for two different values of the persistence of the shocks. For comparison, the same values under no active forward guidance are also plotted. The first result of the exercise to note is that the mean length of periods for which the zero lower bound applies is increasing in \( \gamma_\pi \), as expected under the policy rule (23), for both levels of persistence of the shocks tested. It is notable that forward guidance reduces the variance of inflation, because it tempers large negative inflation responses when the zero lower bound binds. For large values of \( \gamma_\pi \), the variance of inflation across the forecast horizon rises significantly. This reflects two forces: first, the central bank may be holding the interest rate at zero for much longer than is optimal, which increases initial inflation significantly, and second, the volatility caused by shocks while constrained at the zero lower bound.

Interestingly, forward guidance under persistent shocks improves the value of the loss for any value of \( \lambda \) when \( \gamma_\pi \leq 80 \). That is, forward guidance reduces the volatility of inflation and the output gap under the simple forward guidance rule (23). Why? Higher persistence increases the volatility of inflation and output gap fluctuations because the unconditional
variance of shocks increases with persistence. For the Ireland (2004) model, this is reflected in the AD and AS analysis above. Large negative shocks are more likely to cause the zero lower bound to bind, which in turn causes inflation and the output gap to be exposed to stronger adverse effects. Forward guidance then potentially has a more important role in stabilizing inflation and output in response to shocks which push the nominal interest rate to zero. Overall, the simulations suggest that the loss function values fall for values of $\gamma_\pi \leq 120$.

4.8 State-contingent forward guidance

In this section, I distinguish time-dependent forward guidance from state-contingent forward guidance. The latter are policies which announce a path of the interest rate, where the path depends on future realizations of endogenous variables. An example of this type of announcement is a statement indicating that the monetary authority will keep the interest rate at zero until the unemployment rate reaches 6.5%. The Federal Reserve has used such policies, using language which is fairly relaxed about the numerical guides, such as a desire to keep the interest rate low ‘at least as long as as the unemployment rate remains above 6.5%’ (FOMC December 2012 statement). Indeed, since a specified numerical target of an unemployment rate of 6.5% was reached in 2014, the Federal Reserve has not adjusted its short-rate in response, suggesting the numerical targets are more guides to the path of the interest rate rather than triggering thresholds.

State-contingent forward guidance is complicated to model, because the language of such announcements affect current and future outcomes, which in turn affects the speed at which the announced target is reached. That is, achieving the announced threshold is directly dependent on the announced forward guidance period. Shocks that occur during the guidance period complicate the analysis: a properly specified problem would require agents to have beliefs about the possibility that the announced threshold is breached as a result of all possible future paths, and would need to understand how that uncertainty translates into different monetary policy announcements. Under this problem, interest rate projections would affect consumer, firm and monetary policy across the path. The framework developed in this paper for solving linear rational expectations models subject to the zero lower bound is not suited to understanding state-contingent forward guidance and so I focus here on simple time-dependent policy announcements which the Federal Reserve commits to completing.
5 The Federal Reserve during the Great Recession

The results of the previous sections would suggest that the volatility of consumption, output and inflation should have increased in the US since the zero lower bound became binding from 2009. However, the observed volatility of consumption, output and inflation over this period is comparable to decade averages since the 1980s (Table 2). Figure 11 illustrates this. This is somewhat puzzling and suggests the Federal Reserve is using alternative tools to act against shocks when at the zero lower bound. In this section I submit to the algorithm a model which approaches those used in policy analysis, the Smets and Wouters (2007) (SW) model, and study whether announcement policies help reconcile differences between the model predictions and observed series.

![Figure 11 about here.]

5.1 Estimating the Smets-Wouters model over the ZLB period

The SW model consists of optimizing households and firms, a government and a monetary authority. The key micro-founded frictions are the following: sticky nominal price and wages, habit formation in consumption, investment adjustment costs, variable capital utilization and fixed costs in production. There are seven structural shocks: a total factor productivity shock, a risk premium shock and investment-specific technology shock (both of which affect the intertemporal margin), wage and price mark-up shocks (which affect the intratemporal margin), and government spending and monetary policy shocks. Because it is widely used, a full setup of the SW model is omitted here, but the Appendix contains the full set of equations used in estimation.

I use comparable data, sources and transformations as SW up until the end of their estimation period, the fourth quarter of 2004, and append updated data to take the estimation

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<th>Decade</th>
<th>Consumption</th>
<th>Output</th>
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period up to 2014Q2. I replace the policy rate over the period of the zero lower bound, from the first quarter of 2009 onwards, with the series developed by Wu and Xia (2014). They develop an innovative approximation to the Gaussian Affine Term Structure Model (GATSM) which allows a shadow interest rate to be linear in Gaussian factors but which ensures the term structure respects the zero lower bound. This approximation admits an estimate of a model of the term structure subject to the zero lower bound. The shadow interest rate, which equals the short-rate when it is not constrained by the zero lower bound, and which becomes negative when the constraint binds, summarizes the policy actions of the Federal Reserve at the zero lower bound. The intuition for why is the following: when the Fed enacts stimulatory policy at the zero lower bound, it shows up as a lower shadow interest rate because the policy potentially reduces longer-term interest rates. This reduction in longer-term interest rates, and an appropriate estimation of the factors and parameters of the GATSM translates into a reduced shadow interest rate through the linear relationship between the shadow interest rate and the factors. The series is plotted in Figure 12c as the observed series. Wu and Xia (2014) show in an estimated VAR that the series, during the zero lower bound period, implies the same relationships between the interest rate and key macroeconomic variables.

The SW model is estimated on data for two periods: 1966Q1 to 2004Q4 (as in SW) and 1989Q3 to 2014Q2 (to cover the period since the Great Recession) using Bayesian methods. I use the same priors as Smets and Wouters (2007). The estimates are reported in Table 3. The estimated set of parameters for the SW period are close to those reported in Smets and Wouters (2007). The differences are due to data revisions and a different real wage series. Comparing the SW-period estimates with the estimates from 1989Q3 to 2014Q2 reveals that the estimated parameters are similar (a linear fit of a scatterplot of the estimated parameters gives an $R^2$ of 0.9). The similarity of the estimates is reassuring given the use of the shadow interest rate series for 2009Q1 to 2014Q2, suggesting that the shadow interest rate carries the same economic content as movements in the interest rate when the zero lower bound does not bind. This supports the results of Wu and Xia (2014). Some notable differences include the persistence and standard deviation of demand shocks.

Under the estimated set of parameters, the patterns of volatility at the zero lower bound in the SW model are similar to the patterns uncovered in the previous section for the Ireland (2004) model. The volatility of inflation and output growth both increase as the zero lower bound is more likely to bind.
<table>
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5.2 The effect of the zero lower bound

How bad could the Great Recession have been in the absence of Federal Reserve policy? The SW model estimated using the Wu and Xia (2014) shadow interest rate yields estimates of the exogenous shocks and parameter values. Assuming the shadow interest rate approximates the behaviour of the Federal Reserve over the period of the zero lower bound, those estimated shocks are the ‘true’ shocks that the economy is subject to. I construct a counterfactual path of key endogenous variables where the SW economy is subject to the same shocks, but where the zero lower bound is imposed. The counterfactual path asks: if the economy was subject to the same shocks as estimated using the interest rate series that reflects the unconventional monetary policy of the Federal Reserve, but where the zero lower bound was imposed, what would be the implied inflation, output growth and output gap series? The observed and counterfactual paths are illustrated in Figures 12 and 13.

Figure 12 illustrates how the estimated structural shocks together with the constraint of the zero lower bound would imply more volatile output growth series, a weaker inflation profile, a stronger initial decline in output growth, and an increasingly declining output gap. Figure 13 illustrates how output per capita would appear in the counterfactual series where the Federal Reserve is inactive at the zero lower bound. No action by the Federal Reserve would yield an output series which is flat since 2009, at its level reached in 2000. These counterfactual series illustrate the effects of misspecification of the Federal Reserve’s policy during the period of the zero lower bound when estimating a structural model.

5.3 The government spending multiplier

Do exogenous increases in government spending have the same effect when the central bank is actively providing forward guidance policies as compared to when the central bank is inactive? A number of papers have advocated government spending at the zero lower bound because it stimulates significantly the real interest rate through inflation, leading to government spending multipliers above one. I study the effect of active monetary policy at the zero lower bound by calculating the following government spending multiplier, as in Fernández-Villaverde et al. (2012):

\[
\frac{\bar{y}_t - y_t}{\bar{\varepsilon}_{g,1} - \varepsilon_{g,1}},
\]
where \( \tilde{y}_t \) is the value of log output at time \( t \) when exogenous government spending increases by \( \tilde{\varepsilon}_{g,1} - \varepsilon_{g,1} \) at time period 1, and \( y_t \) is log output in the absence of any exogenous increase in government spending. The shock to \( \varepsilon_{g,1} \) is such that \( \tilde{\varepsilon}_{g,1} - \varepsilon_{g,1} = 2\% \), or about the size of the fiscal stimulus in 2009 in the US. This value is computed for three different monetary policy regimes. First, when the central bank is unconstrained in its setting of the nominal interest rate. Second, when a large negative demand shock pushes the nominal interest rate to zero for seven consecutive periods (in the absence of any stimulatory policy from either the central bank or the fiscal authority). And third, after being hit by the same demand shock as in the second regime, the central bank announces it will extend the period that the zero lower bound binds by two quarters, when computed in the absence of any stimulatory policy by the fiscal authority. In terms of the rule specified in (23), this corresponds to a value of \( \gamma_\pi = 10 \) given the scaling factors of the SW model. The multipliers across the path for these three regimes are plotted in Figure 14.

In the SW model, as in other papers, when the zero lower bound binds, the government spending multiplier is larger and greater than one (the red curve) compared to when the central bank is unconstrained and following a Taylor rule (the blue curve). The reasoning lies in the effect that government spending has on the real interest rate. When the zero lower bound binds and the nominal interest rate is fixed at zero, government spending affects inflation and so the real interest rate directly. If the central bank was unconstrained in setting the nominal interest rate, the real interest rate would fall by a smaller amount, because the nominal interest rate would rise in response to the increase in inflation. In the unconstrained case, the stimulatory effect of government spending is offset by the central bank tightening policy, leading to a multiplier which is less than one.

The multiplier falls when the central bank is active in response to the binding zero lower bound. The reason lies in the unkinking of the policy function when the central bank uses forward guidance. An exogenous government spending shock moves the economy up the policy function, but with an unkinked policy function, by less when the central bank is providing forward guidance. Figure 15 provide some intuition. In Drautzburg and Uhlig (2013), an extension of the period of the zero lower bound actually increased the fiscal multiplier. To reconcile this result with the simulations here, the period of forward guidance in this paper is actively set in response to the rate of inflation. Since government spending stimulates inflation, government spending effectively reduces the length of the forward guidance period. The two policies act against each other, reducing the effectiveness, by itself, of fiscal policy.
These results show that it is important to correctly specify what the central bank is doing in response to shocks which drive the nominal interest rate to zero. If it provides additional stimulus through active forward guidance, the role that fiscal policy plays in stimulating the economy changes, becoming less effective.

6 Conclusion

It is difficult to evaluate how the zero lower bound affects economic outcomes and the operation of monetary policy in medium-to-large New Keynesian dynamic stochastic general equilibrium (DSGE) models. This paper addresses this difficulty by developing a piecewise linear approximation to the zero lower bound which is endogenous to the shocks which impact the economy. The approximation is flexible enough to model forward guidance policies and is extremely fast, so that it accommodates how policymakers typically use those models – repeatedly solving the model under different parameter sets or policy options to examine counterfactual paths, run simulations and estimate parameters.

I use the implementation to study the consequences of unanticipated shocks at the zero lower bound. New Keynesian models predict excess volatility in aggregate variables resulting from such shocks. Such volatility is at odds with the experience of the US since 2009. This paper proposes that forward guidance policies at the zero lower bound can help stabilize the effects of shocks when the nominal interest rate is constrained at zero. In an estimated Smets and Wouters (2007) model, I study counterfactual paths of the economy where the Federal Reserve did not use active forward guidance policies, and derive the implied lengths of forward guidance policies which reconcile these counterfactual paths and the observed series. Interestingly, the results predict that agents in the model anticipate a continual extension of an already long period of being at the zero lower bound, starting in 2009Q1.
A Details on solution methods

A.1 Linear rational expectations solution method

The following is adapted from Cagliarini and Kulish (2013). Write a model in matrix form as:

$$\tilde{\Gamma}_0 y_t = \tilde{\Gamma}_1 y_{t-1} + \tilde{C} + \tilde{\Psi} \varepsilon_t, \quad (A.1)$$

where the state vector is defined by and ordered according to:

$$y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ E_t z_{t+1} \end{pmatrix},$$

and where $y_{1,t}$ is an $(n_1 \times 1)$ vector of exogenous and some endogenous variables, and $y_{2,t}$ is an $(n_2 \times 1)$ vector with those endogenous variables for which conditional expectations appear; $z_{t+1}$, $(k \times 1)$, contains leads of $y_{2,t}$; in the model above, however, $z_{t+1} = y_{2,t+1}$ and $k = n_2$. The dimension of $y_t$ is $n \times 1$, where $n = n_1 + n_2 + k$. Also, we assume $\varepsilon_t$ to be an $l \times 1$ vector of serially uncorrelated processes, $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ are $(n_1 + n_2) \times n$ matrices, $\tilde{C}$ is $(n_1 + n_2) \times 1$ and $\tilde{\Psi}$ is $(n_1 + n_2) \times l$.

Because of the presence of expectations, we cannot invert $\tilde{\Gamma}$ and estimate a reduced form version of (A.1). Sims’s (2002) proposal is to append to (A.1) expectations revisions which will be solved as part of the solution. Let $\eta_t$ be the vector of expectations revisions:

$$\eta_t = E_t z_t - E_{t-1} z_t, \quad (A.2)$$

where $E_t \eta_{t+j} = 0$ for $j \geq 1$. For example, if $z_t = y_{2,t}$, then $\eta_t$ are just forecast revisions.

Augment the system defined by Equation (A.1) with the $k$ equations from Equation (A.2) to obtain:

$$\Gamma_0 y_t = C + \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \quad (A.3)$$

where the matrices $\Gamma_0, \Gamma_1, C, \Psi$, and $\Pi$ are of conformable dimensions. $\Gamma_0$ is now an $n \times n$ matrix, which we will invert with a Schur (QZ) decomposition, and impose conditions such that we can remove the $\eta_t$ from the system.

To solve (A.3) as Sims (2002), take a Schur (QZ) decomposition of $(\Gamma_0, \Gamma_1)$ to get:

$$Q' \Lambda Z' = \Gamma_0$$
$$Q' \Omega Z' = \Gamma_1,$$
where $\Lambda$ and $\Omega$ are both upper triangular. The Schur decomposition gives that $QQ' = I$ and $ZZ' = I$. A property of the decomposition is that the generalized eigenvalues of $(\Gamma_0, \Gamma_1)$ are ratios of diagonal elements of $\Omega$ and $\Lambda$. Pre-multiply model equation by $Q$ and define $w_t = Z'y_t$ to rewrite the system as:

$$\Lambda w_t = \Omega w_{t-1} + Q(C + \Psi \varepsilon_t + \Pi \eta_t).$$

Define $w_{1,t} = Z'_1 y_t$ and $w_{2,t} = Z'_2 y_t$. $\Lambda$ and $\Omega$ are upper triangular - rearrange the system so that the explosive eigenvalues correspond to the lower right blocks of $\Lambda$ and $\Sigma$, partitioning $w_t$ and rewriting the system as:

$$
\begin{pmatrix}
\Lambda_{11} & \Lambda_{12} \\
0 & \Lambda_{22}
\end{pmatrix}
\begin{pmatrix}
w_{1,t} \\
w_{2,t}
\end{pmatrix}
=
\begin{pmatrix}
\Omega_{11} & \Omega_{12} \\
0 & \Omega_{22}
\end{pmatrix}
\begin{pmatrix}
w_{1,t-1} \\
w_{2,t-1}
\end{pmatrix}
+
\begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix}
(C + \Psi \varepsilon_t + \Pi \eta_t).
$$

The lower block of the system are those equations which correspond to the $m$ explosive generalised eigenvalues of $(\Gamma_0, \Gamma_1)$. The lower set of equations are not affected by $w_{1,t}$. Isolate these:

$$\Lambda_{22} w_{2,t} = \Omega_{22} w_{2,t-1} + Q_2 (C + \Psi \varepsilon_t + \Pi \eta_t).$$

For stability of the system, we need $\eta_t$ to offset the effect of $\varepsilon_t$ on $w_{2,t}$. For example, if $C = 0$, we need:

$$Q_2 \Psi \varepsilon_t = -Q_2 \Pi \eta_t. \quad (A.4)$$

Equation (A.4) says that stability depends on the expectations revisions $\eta_t$ offsetting the effect of fundamental shocks $\varepsilon_t$ on $w_{2,t}$. Expectations revisions then ensure that the system is placed on the saddle path to stability.

Sims (2002) shows that a solution to the above system exists, and is unique, if the number of explosive eigenvalues of $(\Gamma_0, \Gamma_1)$, $m$ equals the number of variables which appear as expectations in the system, $k$. Under this condition, the system is on a saddle path to a steady-state from any initial condition. (Note, there are weaker conditions just for stability.) If this is true, and if the solution is stable then there is a matrix $\Phi$ such that:

$$Q_1 \Pi = \Phi Q_2 \Pi.$$ 

30
By premultiplying the system by \([I_{n-p}, -\Phi]\), the coefficient on \(\eta_t\) is:

\[
Q_1 \Pi - \Phi Q_2 \Pi.
\]

Since existence of a solution requires \(Q_1 \Pi = \Phi Q_2 \Pi\), the \(\eta_t\) drop out of (A.1), so that a solution to the model can be written as:

\[
y_t = S_0 + S_1 y_{t-1} + S_2 \varepsilon_t + S_y \mathbb{E}_t \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j},
\]

where:

\[
H = Z \begin{pmatrix}
\Sigma_{11}^{-1} & -\Sigma_{11}^{-1} (\Sigma_{12} - \Phi \Sigma_{22}) \\
0 & I
\end{pmatrix},
\]

\[
S_0 = H \begin{pmatrix}
Q_1 - \Phi Q_2 \\
(\Sigma_{22} - \Omega_{22})^{-1} Q_2
\end{pmatrix} C,
\]

\[
S_1 = H \begin{pmatrix}
\Omega_{11} & \Omega_{12} - \Phi \Omega_{22} \\
0 & 0
\end{pmatrix},
\]

\[
S_2 = H \begin{pmatrix}
Q_1 - \Phi Q_2 \\
0
\end{pmatrix} \Psi,
\]

\[
S_y = -H \begin{pmatrix}
0 \\
I_m
\end{pmatrix}.
\]

The solution (A.5) is in the desired \(\text{VAR}(1)\) form.

### A.2 Foreseen structural changes

The structural parameters of the economy are known to change into the future. In particular, suppose the economy is expected to evolve with the following structure: in time period \(t = 1\), the economy starts with the following structure:

\[
\tilde{\Gamma}_0,1 y_t = \tilde{\Gamma}_{1,1} y_0 + \tilde{C}_1 + \tilde{\Psi}_1 \varepsilon_1,
\]

and in time periods \(2 \leq t \leq T\), the structural parameters of the economy evolve according to:

\[
\Gamma_0,1 y_t = \Gamma_{1,1} y_{t-1} + C_t + \Pi \eta_t + \Psi_t (\varepsilon_t^u + \varepsilon_t^a),
\]

where \(\varepsilon_t^u\) are shocks which are unanticipated at time period \(t = 1\), \(\varepsilon_t^a\) are shocks which are anticipated at \(t = 1\). Notice that expectations revisions are included as the system evolves. Unanticipated shocks are added to show that it is possible to solve the model subject to foreseen structural changes and unanticipated shocks, though the solution would need to be computed each time period. Also notice that the matrices specifying the structural
parameters are time-varying. After time period \(T + 1\), the structural parameters of the economy are fixed, so that the system becomes:

\[
\bar{\Gamma}_0 y_t = \bar{\Gamma}_1 y_{t-1} + \bar{C} + \bar{\Pi}\eta_t + \bar{\Psi}\varepsilon_t.
\]

Stacking \(T \times (n_1 + n_2 + k) + \tilde{m} - k\) equations yields:

\[
\begin{bmatrix}
\bar{\Gamma}_{0,1} & 0 & \ldots & \ldots & 0 \\
-\Gamma_{1,2} & \Gamma_{0,2} & \ddots & \ddots & \vdots \\
0 & -\Gamma_{1,3} & \Gamma_{0,3} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & -\Gamma_{1,T} & \Gamma_{0,T} \\
0 & \ldots & \ldots & 0 & \tilde{Z}'_2
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_T
\end{bmatrix}
= \begin{bmatrix}
\tilde{C} + \bar{\Gamma}_{1,1} y_0 + \bar{\Psi}_1 \varepsilon_1^a \\
C_2 + \Psi_2 \varepsilon_2^a \\
\vdots \\
C_T + \Psi_T \varepsilon_T^a \\
\tilde{\omega}_{2,T}
\end{bmatrix}.
\]

More concisely:

\[A\mathbf{y} = \mathbf{b}.\]

The necessary condition to invert \(A\) is that the final (bar) structure of the economy has a solution, which ensures the economy reaches its saddle path eventually. Given a unique solution to the final (bar) structure, Cagliarini and Kulish (2013) show that the intermediate path of the economy is unique. Practically, \(\mathbf{y} = (y'_1, \ldots, y'_T)\) is a \(T \times (n_1 + n_2 + k)\) vector. As discussed in Sims (2002) and in Section A.1, a solution to the final system implies \(\tilde{m} = k\). This condition implies \(A\) is a square matrix. An additional condition is that \(A\) be full rank. This is generally the case unless perverse parameters are used. The system can (mostly) be unconstrained in the intermediate stage. If the system is on a saddle path eventually, there is (usually) a unique path.

**B  Proof of Results 1**

[TBD] Take solution for final period: \(y_T = S_0 + S_1 y_{T-1}\). Isolate the law of motion for the expectational part. Plug that in with \(E_t y_{t+1} = y_{t+1}\). This is the same matrix as in Cagliarini and Kulish (2013). Then solve the matrix backwards. This is the approach of Guerrieri and Iacoviello (2014).
C Non-linear solution of Ireland (2004) model

C.1 Algorithm

I first discretize \( a_t \) with a Tauchen approximation described in the next subsection. The solution will be in terms of two functions \( f_1(y_{t-1}, a_t) \) and \( f_2(y_{t-1}, a_t) \) which approximate the expectations given by \( \frac{1}{\beta} \frac{\alpha}{\sigma} y_t \) and \( \frac{\alpha}{\sigma} y_t^{\theta - 1} + \phi \left( \frac{\pi}{\pi} - 1 \right) \pi \). First, I constrain \( y_{t-1} \) to lie between 0.95 and 1.05 times the steady-state value of \( y \). Together with the discretized \( a_t \), this gives a grid \( \{ y, a \} \). I use the guess-and-verify method to solve for the policy functions. The algorithm is:

1. Guess the values of \( f_1(s^i) \) and \( f_2(s^i) \) at each point \( s^i \in \{ y, a \} \). Call these guesses \( \hat{f}_1(s^i) \) and \( \hat{f}_2(s^i) \).

2. Using the guesses \( \hat{f}_1(s^i) \) and \( \hat{f}_2(s^i) \):
   - for each \( i \) (for each state):
     - Guess \( \pi_t \). Using the guess for \( \pi_t \):
       - (a) Determine the ratio \( \frac{\alpha}{\gamma_t} \).
       - (b) Using the ratio \( \frac{\alpha}{\gamma_t} \) and the guess \( \hat{f}_2(s^i) \), obtain \( y_t \).
       - (c) Using \( y_t \), obtain \( c_t, a_t, g_t \) and then \( r_t \), where \( r_t \) is calculated subject to the zero lower bound.
       - (d) Using \( \hat{f}_1(s^i) \) and \( r_t \), compute the implied consumption and call it \( \tilde{c}_t \).
         - Check the computed \( \tilde{c}_t \) against \( c_t \). Update guess of \( \pi_t \) until \( |\tilde{c}_t - c_t| \) converges.

3. With the equilibrium policy functions at time \( t \), compute the expectations:
   \[
   \mathbb{E}_t \left[ \frac{a_{t+1}}{c_{t+1}} \frac{1}{\pi_{t+1}} \right] \quad \text{and} \quad \mathbb{E}_t \left[ a_{t+1} \frac{y_{t+1}}{c_{t+1}} \left( \frac{\pi_{t+1}}{\pi} - 1 \right) \frac{\pi_{t+1}}{\pi} \right],
   \]
   using the transition matrix for the discretized \( \theta_t \). Adjust the guesses \( \hat{f}_1(s^i) \) and \( \hat{f}_2(s^i) \) until convergence.

The algorithm yields policy functions for all the endogenous variables.

C.2 Tauchen method

Suppose we have the process \( y_t = \rho y_{t-1} + \varepsilon_t \) with \( |\rho| < 1 \) and \( \varepsilon_t \) is normally distributed with variance \( \sigma^2 \). Tauchen’s method is used to give a discrete approximation to this continuous
process. Let $\varepsilon_t$ be drawn from a distribution $G$. Standardize the distribution and denote the standardized distribution by $F$. Let $\bar{y} = \{\bar{y}_1, \ldots, \bar{y}_N\}$ be the discrete process which approximates $y$. Choose $\bar{y}_N$ which is a multiple $m$ of the unconditional standard deviation:

$$\bar{y}_N = m \left( \frac{\sigma_\varepsilon^2}{1 - \rho^2} \right)^{\frac{1}{2}}. $$

Let $\bar{y}_1 = -\bar{y}_N$ for $G$ symmetric and distribute the intermediate points evenly between. Let $d$ be the distance between successive points. The transition probability from state $j$ to state $k$ can be written:

$$\pi_{jk} = \Pr \{ \bar{y}_t = \bar{y}_k | \bar{y}_{t-1} = \bar{y}_j \}$$

$$= \Pr \left\{ \bar{y}_k - \frac{d}{2} - \rho \bar{y}_j \leq \varepsilon_t \leq \bar{y}_k + \frac{d}{2} \right\}$$

$$= \Pr \left\{ \bar{y}_k - \frac{d}{2} - \rho \bar{y}_j \leq \varepsilon_t \leq \bar{y}_k + \frac{d}{2} - \rho \bar{y}_j \right\},$$

so that, for $1 < k < N - 1$, for each $j$ choose:

$$\pi_{jk} = F \left( \frac{\bar{y}_k + d/2 - \rho \bar{y}_j}{\sqrt{\sigma_\varepsilon}} \right) - F \left( \frac{\bar{y}_k - d/2 - \rho \bar{y}_j}{\sqrt{\sigma_\varepsilon}} \right),$$

and at the boundaries:

$$\pi_{j1} = F \left( \frac{\bar{y}_1 + d/2 - \rho \bar{y}_j}{\sqrt{\sigma_\varepsilon}} \right) \quad \text{and} \quad \pi_{jN} = 1 - F \left( \frac{\bar{y}_N + d/2 - \rho \bar{y}_j}{\sqrt{\sigma_\varepsilon}} \right).$$

D  Smets and Wouters model

E  Data appendix
References


Jing Cynthia Wu and Fan Dora Xia. Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound. manuscript, 2014.
Figure 1: Impulse response to a negative five standard deviation demand shock

(a) Interest rate

(b) Inflation

(c) Output growth
Figure 2: Iterations of algorithm

(a) Iteration 1: No ZLB
(b) Iteration 2
(c) Iteration 5
(d) Iteration 10
Figure 3: Stochastic path, ZLB algorithm versus OccBin algorithm
Figure 4: Initial response of inflation to $a_t$ shocks

(a) Nominal interest rate

(b) Inflation

(c) Growth rate
Figure 5: Standard Deviation of interest rate, inflation and output growth

Note: The standard deviations of the interest rate, inflation and output growth are normalized by their respective standard deviations for $\varepsilon_a = 0.08$. 
Figure 6: Fancharts for unanticipated shocks each period

(a) Nominal interest rate

(b) Inflation

(c) Growth rate
Figure 7: Initial response of variables to $a_t$ shocks with forward guidance

(a) Nominal interest rate
(b) Initial forward guidance length
(c) Inflation
(d) Growth rate
Figure 8: Impulse response to a negative six standard deviation demand shock

(a) Interest rate

(b) Inflation

(c) Output growth
Figure 9: Loss function calculations, $\rho_a = \rho_\theta = 0.2$

(a) Standard deviation of $\hat{\pi}_t$

(b) Standard deviation of $\hat{x}_t$

(c) Loss value

(d) Mean length of zero lower bound periods
Figure 10: Loss function calculations, $\rho_a = \rho_\theta = 0.8$

(a) Standard deviation of $\hat{\pi}_t$

(b) Standard deviation of $\hat{x}_t$

(c) Loss value

(d) Mean length of zero lower bound periods
Figure 11: Inflation, output growth and consumption growth
Figure 12: Observed series and counterfactual (under ZLB) estimate

(a) Inflation $\hat{\pi}_t$

(b) Change in output $\hat{d}_t$

(c) Nominal interest rate $\hat{r}_t$
Figure 13: Observed series and counterfactual (under ZLB) estimate (cont.)

(a) Output gap $\hat{x}_t$

(b) Index of output

(c) Length of ZLB periods
Figure 14: Government spending multiplier

Note: Active policy implies the zero lower bound period is extended by two periods, and all agents anticipate this change.
Figure 15: Policy functions under inactive policy and forward guidance (dashed)

(a) Policy functions: inflation

(b) Policy functions: output growth