VOLATILITY INDICES AND STATE-PREFERENCE PRICING

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Abstract

This paper is the first to use a state-preference pricing approach with Black-Scholes analytic second derivatives to develop a forward-looking volatility index (FIX), as a forecast of the next 30-day market risk-neutral volatility. Using S&P 500 index (SPX) option prices from 1996 to 2010, we find that FIX is 99% correlated with the current CBOE volatility index (VIX) and it is a better estimator for the short-term realized volatility of SPX returns than VIX. Our result is robust to different measures of realized market volatilities. We argue that VIX is not a model-free measure and due to the potential noise issues from the thinly traded deep out-of-the-money (OTM) options, VIX may over-estimate the future volatility. This is supported by our results. Moreover, unlike VIX which is affected by the availability of strike prices and may be manipulated by trading deep OTM put options, we show that FIX is more difficult for manipulation and less data dependent. We also show that FIX provides a better volatility forecast than other alternative measures, such as the squared root of GARCH (1, 1) variance. Our results reinforce previous findings in the literature that at-the-money Black-Scholes implied volatility is an efficient and more superior forecast of subsequent realized volatility.
This thesis develops a forward-looking risk-neutral volatility index (FIX) and a skewness index (SIX), as alternatives to the current CBOE Volatility Index (VIX) and Skewness Index (SKEW), respectively. Our risk-neutral measures are constructed using a state-preference pricing approach with Black-Scholes analytic second derivatives as the state prices.

VIX has gained increasing attention from investors since its introduction in 1993. It has often been referred as the fear index. CBOE revised the definition and formulation of VIX in 2003. The new VIX is known as a model-free measure and it is based on the concept of the fair value of future variance, as proposed in Demeterfi et al. (1999).

SKEW has been published by CBOE as a complementary measure of tail risk to VIX since 2011. The methodology of SKEW originates from the seminar paper by Bakshi, Kapadia, and Madan (2003). SKEW has faced criticisms that it has no clear empirical relationship with the market performance.

The first essay serves as the cornerstone of this thesis. We show that the new VIX is developed in the diffusion framework and it is not a model-free measure. We argue that the inclusion of all thinly traded out-of-the-money (OTM) S&P 500 index (SPX) options may add more noises in predicting the future realized volatility. Furthermore, the ignorance of trading volume of these OTM options may open for potential manipulation of VIX. This motivates us to construct an alternative volatility forecast, FIX, by using the state-preference pricing approach.

The second essay serves as an extension of the first one. The motivation comes from the fact that VIX, as a volatility measure that considers deviations on both sides of the mean, fails to distinguish between a potential downside threat and an upside gain. We decompose FIX into a forward-looking lower partial moment volatility index as a forecast for market downturn, which we denote the bear index (BEX); and an upper partial moment counterpart, which we denote the bull index (BUX).

The third essay constructs the market symmetric index (SIX) by using the ratio of BEX to BUX. We show that SIX has some predictive power on the future market returns and short-term market realized skewness, which are not captured by SKEW. This makes SIX to be an important indicator of institutional anxiety regarding stock market uncertainty, as a complement to the existing VIX.

The thesis will take the following structure:
Chapter I: Introduction
Chapter II: Volatility Indices and State-Preference Pricing
Chapter III: Partial-Moment Volatility Indices
Chapter IV: Market Symmetric Index
Chapter V: Evidence in DJIA and NASDAQ 100
Chapter VI: Conclusion

This paper is the Chapter II of this thesis.
1 Introduction

Volatility is the key ingredient in the financial modeling. It occupies a prominent role in both financial theory and empirical implementation. It is most frequently referred to the standard deviation of the return distribution. Like the price of any financial asset, volatility varies stochastically through time. On the other hand, unlike the asset price which is updated by the latest bids and asks in the market, volatility is an inherently unobserved variable.

The past four decades have seen volatility forecasting as one of the most active and successful areas of research in finance and economic. It has been discussed in a variety of contexts. These include, but not limited to, historical volatility, ARCH/GARCH volatility and implied volatility. We refer to surveys by Poon and Granger (2003) and Andersen, Bollerslev, Christoffersen, and Diebold (2005) for a more comprehensive coverage on the existing literature.

Historical volatility is widely used as a simple approach to quickly assess future price movements. It is computed as the sample standard deviation of returns over some specific time period in the past. There is a trade-off between computing based on a longer or shorter history, of which the former may reduce the relevance for the future movements and the latter may create more noise (Engle, 2003).

A more popular approach is the autoregressive conditional heteroskedasticity, better known as ARCH model (Engle, 1982). ARCH model uses weighted average of past squared forecast errors to describe future variance. An important extension, namely the generalized autoregressive conditional heteroskedasticity or GARCH model, is first developed by Bollerslev (1986) and also discussed independently by Taylor (1986). The popular GARCH(1,1) variance is essentially a weighted average of three components, including a constant long run average variance, a forecast variance from previous period and an additional information term that has not been captured by the second term.

Implied volatility is fundamentally different from the above estimates. It is typically implied from option prices. Since the core component for pricing an option is the probability to be exercised at the maturity date, the traded option price should therefore reflect market participants’ beliefs on the underlying asset price movements during the life of the option.

This original Volatility Index (VXO), published by CBOE since 1993, is an average of implied volatilities from eight at-the-money (ATM) call and put S&P 100 index (OEX) options. Since its introduction, it has gained increasing attention from investors and has become the benchmark measure for stock market volatility. Unlike other price indices (such as S&P 500) that reflect the instantaneous changes in the equity market, VXO measures the expected volatility for the next 30-day period.
CBOE revised the formulation of VIX in 2003. The new VIX extracts the market expectations of future price movements from traded S&P 500 index (SPX) options. It can be seen as a summation of all out-of-the-money (OTM) call and put options, each weighted by the reciprocal of the squared strike prices. Since the criterion for being included in the pool of OTM options for VIX calculation ignores the trade volume, this leads to two potential concerns. First, VIX may be manipulable by bidding on OTM options without having to exercise a trade. Second, the inclusion of thinly traded OTM options may create more noises in predicting the future volatility.

This motivates us to develop an alternative volatility measure. In this paper, we use a state-preference pricing approach with Black-Scholes analytic second derivatives to develop a forward-looking volatility index (FIX). Since we use the average of implied volatilities from four ATM call and put options on S&P 500 index as the key input in the state price, FIX is much harder for manipulation than VIX. Moreover, our study on FIX forecast ability can be seen as a joint test on both state-preference pricing framework and the efficiency of implied volatility as a proxy for future realized volatility.

The remainder of this paper is organized as follows. Section 2.2 reviews the evolution of CBOE volatility index VIX and discusses drawbacks with VIX. Section 2.3 describes the data and the methodology used to construct FIX. Section 2.4 studies the performance of FIX as a volatility forecast. Section 2.5 concludes.

2 CBOE VIX: History and Drawbacks

2.1 Old VIX

CBOE started publishing the VIX, also known as the fear index Whaley (2000), on a real-time basis throughout each trading day in 1993. It soon became the benchmark measure for stock market volatility. VIX is first introduced in a paper by Whaley (1993). In 2003, CBOE revised the methodology for the calculation of VIX and since then the ticker for the original volatility index is renamed to VXO. CBOE continues to provide quotes of VXO on a daily basis.

The construction of VXO uses OEX options data. It is computed as an average of the Black and Scholes (1973) option implied volatilities on eight ATM call and put options at the two nearest maturities (or the next two nearest maturities if the nearest maturity is within eight calendar days). That is, at each maturity, CBOE selects 1 pair of call and put option that straddles the spot OEX level and another pair that is nearest to the spot level. Average implied volatilities from these two pairs are interpolated to obtain the ATM spot implied volatility. They are further interpolated to obtain a 22-trading day volatility. For the details of the
VXO computation, we refer to the original set up in Whaley (1993).

While the annualization of the Black-Scholes implied volatility is based on an actual/365 day-counting convention, CBOE uses a special trading-day convention that consequently induces an artificial upward bias. This makes VXO less favorable as a candidate for the comparison with annualized realized volatility computed from index returns (Carr & Wu, 2006).

2.2 New VIX

CBOE revised the definition and formulation of VIX on September 22, 2003. There are three major changes involved.

First, the construction of VIX shifts to SPX options data. SPX options are European style, which make them relatively easier to price, in comparison to the American style OEX options. As presented in Whaley (2009), the SPX option market has become the dominant index option market in the U.S. with the trade volume of SPX options being about 12.7 times of OEX ones. In addition, over the past two decades SPX has become a more widely followed market index than OEX, as reflected in both the ETF trade volumes and the media coverage.

This switch in the underlying market index has no real impacts on the return/risk properties from investors’ perspective. With near identical levels of mean daily returns and slightly different standard deviations of daily returns from 1986 to 2008, OEX and SPX markets are highly correlated. That is, trading OEX and SPX options should be equally effective from a risk management’s perspective (Whaley, 2009).

Second, the methodology for computation of VIX shifts from a model-based approach to a model-free one. The principal of new VIX is built on the fair value of future variance developed by Demeterfi, Derman, Kamal, and Zou (1999) (DDKZ thereafter). More specifically, DDKZ show how the dynamic hedging of a log contract can capture realized volatility. This concept is originated from the pioneering work of Breeden and Litzenberger (1978). Jiang and Tian (2007) show that the fair value of future variance is mathematically equivalent to the model-free implied variance independently developed by Britten-Jones and Neuberger (2000). Derivation of DDKZ’s approach and its linkage to the new VIX formula is briefly presented in the appendix.

Third, rather than restricting on sourcing from ATM index options, the new VIX uses near-term and next-term OTM call and put SPX options with at least 8 days to expiration. There are two criterion for being included in the pool of OTM options for VIX calculation. Firstly, the option must have a non-zero bid price. Secondly, the option must not be beyond two consecutive strike prices with zero bid prices1.

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1As outlined in CBOE (2009), for an OTM put option, one starts with the first put strike
At each maturity, the following generalized formula is applied to obtain the future variance estimate:

$$\sigma_j^2 = \frac{2}{T} \sum_i \frac{\Delta K_i e^r Q(K_i)}{K_i^2} e^{T(K_i)} - \frac{1}{T} \left[ \frac{F}{K_0} - 1 \right]^2$$  \hspace{1cm} (1)

where $T$ is the time to expiration\(^2\), $F$ is the forward index level for maturity $T$ derived from SPX option prices, $K_i$ is the strike price of $i^{th}$ OTM option, $\Delta K_i$ is the strike price interval\(^3\), $r$ is the risk-free interest rate to expiration, $Q(K_i)$ is the average of the highest bid price and lowest ask price for each option at $K_i$ and $K_0$ is the strike price immediately below the forward index level $F$ (CBOE, 2009).

$\sigma_j$’s from two maturities are inter- or extra-polated to yield a 30-day measure of the expected volatility of SPX market, as presented in equation (2).

$$\text{VIX} = 100 \times \sqrt{\left\{ T_1 \sigma_1^2 \left[ \frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}} \right] + T_2 \sigma_2^2 \left[ \frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}} \right] \right\} \frac{N_{365}}{N_{30}}}$$  \hspace{1cm} (2)

where $N_x$ denote the number of minutes to some future date $x$ and $T_1 < T_2$. This time interpolation implies the new VIX represents an annualized volatility using the actual/365 day-counting convention. It is worth noting that it no long suffers from the artificial upward bias criticism received in the VVO calculation (Carr & Wu, 2006).

The switch from a model-based to a model-free approach enables one to replicate the payoff of the VIX futures and options contracts. Indeed, CBOE launched trading of VIX futures contracts in May 2004 and VIX option contracts in February 2006 on the CBOE Futures Exchange. The latter has gained increasing popularity as a new risk management tool.

### 2.3 Drawbacks with New VIX

Despite the more profound theoretical underpinning of the new VIX, there are a number of potential drawbacks worth discussing.

First, VIX is not a model-free measure of future volatility. As shown in the derivation provided in the appendix, the underlying assumption of the revised VIX methodology is that the asset price dynamic follows the geometric Brownian motion below the ATM index level and moves to successively lower strike prices. Once two put options with consecutive strikes prices have zero bid prices, then no put options with lower strike prices are included in the VIX calculation. Similar analogy applies for OTM call options.

\(^2\)Since August 24, 1992, SPX options expire at the open on the third Friday of the contract month. $T$ is the summation of minutes remaining until midnight of the current day, total minutes in the calendar days between current day and settlement day and minutes from midnight until 8:30 a.m. (Chicago Time) on SPX settlement day.

\(^3\)\(\Delta K_i = (K_{i+1} - K_{i-1})/2\) or $\Delta K_i = K_{2^{nd} \text{ lowest}} - K_{\text{lowest}}$ or $\Delta K_i = K_{\text{highest}} - K_{2^{nd} \text{ highest}}$. 

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Furthermore, this diffusion framework does not allow for potential jumps in the equity return.

Second, there are several types of approximation errors induced by the transformation from DDKZ’s fair value of future variance formula to VIX formula. As examined in a study by Jiang and Tian (2007), these approximation errors include discretization and truncation errors. Discretization errors are induced by approximating the numerical integrations in DDKZ with discretized summations. Truncation errors are induced by truncating the infinity range of strike prices assumed in DDKZ to a finite range restricted by the limited trading data. Since the lowest and highest strike price varies over time, the truncation error may change substantially. Jiang and Tian (2005) show that these errors are negligible if the downside and upside limits span at least three standard deviations from the ATM level. Thus, it may be reasonable to assume the truncation error to be less prominent when the SPX market is more volatile, where deeper-OTM options are more heavily traded.

Third, the linear interpolation of near and next term maturities to form a fixed 30-day measure may induce an error, if the model-free implied variance does not follow a linear function of maturity (Jiang & Tian, 2007). By examining the term structure of volatility implied in foreign exchange options, Xu and Taylor (1994) show that the term structure is neither linear nor monotonic.

Fourth, CBOE uses the average of the lowest ask price and the highest bid price as a proxy for the option price. As documented in the literature (e.g., Figlewski (2010)), bid-ask spreads are quite wide for options that are less traded. The midpoint of wide spread may bias the true information about the implied volatility at that particular strike price level. This impact is more substantial for OTM SPX calls than OTM puts, where the latter is more heavily traded as portfolio insurances.

Fifth, a close examination of VIX formula in equation (1) reveals that an OTM put option receives more weights than an OTM call option, even if at the same moneyness level. In the original set up of DDKZ, where an infinity amount of strike prices are available, this problem is less of a concern because an OTM put option is cheaper than an equally OTM call option in the standard Black model (Carr & Lee, 2009).

Sixth, the criterion for being included in the pool of OTM options for VIX calculation does not consider the trade volume. As mentioned in the previous section, an OTM option is selected as long as it has non-zero bid price and not beyond two consecutive strike prices with zero bid prices. That is, a possible VIX manipulation strategy may involve placing higher bids (to increase the average of bid-ask spread)

\[ \frac{1}{K^2} \]

By definition, the strike price of an OTM put option is lower than an OTM call option. Combined with the fact that the weight \( \frac{1}{K^2} \) in equation (1) is a decreasing function of \( K \), more weights have been placed on OTM put than OTM call options.
across a range of OTM options, or even placing bids in the deep OTM options (that originally have zero bids) for them to be updated in the VIX calculation. This strategy may be more effective by trading OTM put options, because mechanically higher weights are allocated to OTM puts.

In this paper, we do not attempt to address all the aforementioned problems and improve on the new VIX. Instead, we propose an alternative measure FIX, which is based on completely different methodology, and examine whether FIX outperforms SKEW as a forward-looking risk-neutral volatility measure.

3 FIX

3.1 Data Collection

In the previous section, we discuss the potential drawbacks of the new VIX. To fix these problems with VIX, we develop a forward-looking volatility index, FIX, as a proxy for market realized volatility by using the state-preference pricing approach with Black-Scholes analytic second derivatives. FIX is constructed on each trading day from January 4, 1996 to October 29, 2010.

Daily SPX option quotes are obtained from OptionMetrics, which provides historical prices of options based on closing quotes at the CBOE. The data set includes the highest closing bid and ask quotes for each SPX option and the Black-Scholes implied volatilities based on the average of best bid and ask. The option price is approximated by the average of best bid and ask. This is a standard approach adopted by CBOE and other researchers.

We use the US one-month and three-month Treasury-bill yields, adjusting for the dividend yields, as the risk-free interest rates. Treasury-bills and dividend yields are obtained from the Federal Reserve Bulletin and Option Metrics, respectively. For SPX option with the maturity shorter (longer) than either of the two Treasury-bill maturities, the risk-free rate is approximated by the dividend-adjusted one-month (three-month) yield.

Daily quotes of VIX are obtained from both OptionMetrics and CBOE VIX Historical Price Data to ensure the completeness of the data.

To select the final sample of SPX options, we apply several commonly used data filters from both academia (e.g., Jiang and Tian (2007); Carr and Wu (2009)) and industry (e.g., (CBOE, 2009)). It is worth noting that although we only need near-
the-money options that straddle the 30 days maturity (or 22 working days), we use the entire final sample to examine whether we can replicate VIX in order to eliminate any potential data issue.

1. Only SPX options with maturities range from 7 to 81 days are included in the sample. Very short-dated contracts are eliminated to avoid any market micro-structure impacts, liquidity issue and price effects from trading strategies related to contract expiration (Figlewski, 2010). We also exclude very long maturity contracts to minimize computation time, due to the fact that options with maturity longer than 81 days are not possible to be selected as either a near- or next-option in our sample.

2. Options with bid prices less than $0.05 are eliminated. In addition, at each trading day, any call (put) option with strike price higher (lower) than that of the first two consecutive calls (puts) with zero bid prices are excluded, even if the corresponding bid prices are greater or equal to $0.05.

3. ITM options are generally more illiquid than ATM or OTM options (Aït-Sahalia & Lo, 1998). Hence any call (put) options with strike prices below (above) or equal to the forward index level are eliminated. This forward index level is computed as the index level at the end of each trading day growing at the risk-free rate defined above for a 30-day period.

4. We apply the put-call parity on each remaining option and exclude any that appears to be mis-priced.

5. Options with Black-Scholes implied volatilities below 0 or above 1 are removed from the sample.

### 3.2 State-Preference Pricing Framework

There are three basic constraints in asset pricing models, namely the absence of arbitrage, single-agent optimality and market equilibrium. The unifying implication from these three constraints is the existence of a state price, also known as a positive discount factor, for each future state, such that the price of any asset is the state price weighted sum of its future payoffs (Duffie, 2001). This concept originates from the general equilibrium model of security markets in Arrow and Debreu (1954).

Arrow (1964) and Debreu (1959) propose that the fundamental value of any financial asset should be the sum of future payoffs multiplied by state prices. This infers that, given the market price of an asset and its future payoffs, one can estimate the fair discount rate to match up the fundamental value with the observed price.

In the state-preference framework, uncertainty is modeled by assuming that there is a number of different states $s = \{1, \ldots, S\}$ in each future period of the world. Each investor has their own assessment of the probability that a particular state will
occur in a given time period. The main result in this framework is that the price of a security can be expressed as a simple function of the payoff in each state and the corresponding state price. This is principally similar to the result proposed in Harrison and Kreps (1979) and Harrison and Pliska (1981).

In a complete market setting, the stochastic discount factor can capture investors’ marginal rate of substitution between the current consumption and consumption in some future state \( s \) at time \( t \). Hence one may argue that, with the given market price of a call or put option, the entire risk-neutral distribution of the underlying asset can be extracted. As discussed in Ross (1976), even simple options can span the whole state space and thus the price of any option contains information on state prices.

A substantial amount of literature (e.g., Merton (1973), Breeden (1979) and Cox, Ingersoll, and Ross (1985)) incorporates state prices in theoretical models in the modern asset pricing theory. This stream of research is lead by the pioneering study by Breeden and Litzenberger (1978). They show that the price of an elementary security, or the state price, may be modeled as the second derivative of a call or a put option price. Building on this work, Rubinstein (1994, 1998) proposes an alternative method to infer state-contingent prices from option prices. Similar extensions are done by Derman and Kani (1994, 1998) and Derman, Kani, and Chriss (1996).

The breakthrough in this area is by Britten-Jones and Neuberger (2000). They are the first one to show that, under diffusion assumptions, options prices fully specify the risk-neutral integrated return variance between the current date and some future date. We refer to Jiang and Tian (2005) and Carr and Wu (2009) for a recent discussion on the application of this approach.

The basic form of the state-preference pricing equation is:

$$P_t = \sum_{s=1}^{S} (\Phi_{s,t+1} d_{s,t+1})$$

where summing over all possible states \( \{1, \ldots, S\} \), \( P_t \) is the price of some asset at time \( t \), \( \Phi_{s,t+1} \) is the state price and \( d_{s,t+1} \) is the payoff of this asset at state \( s \) and time \( t + 1 \).

It is preferable to look at the forward-looking variance \( \text{FIX}^2 \). This is similar to the process involved in the theoretical derivation of VIX that they firstly replicate the fair value of future variance. Once \( \text{FIX}^2 \) is obtained, it is straight-forward to compute \( \text{FIX} \) as its squared-root.

\( \text{FIX}^2 \) can be viewed as a financial asset that pays a dollar amount of \( (\ln(K_s/S_0))^2 \) at some future date \( T \) for every future index level \( K_s \) and spot price level \( S_0 \).
Following the state-preference pricing approach, $\text{FIX}^2$ is defined as:

$$\text{FIX}^2 = \sum_{s=1}^{S} \Phi_s \left( \ln \left( \frac{K_s}{S_0} \right) \right)^2$$  \hspace{1cm} (4)

Here $S_0$ is the current SPX level, $K_s$ is the $s^{th}$ option with strike $K_s$, $T$ is the time to maturity for the selected option\(^7\) and $\Phi_s$ is state price (or equivalently the risk-neutral probability in this context) at strike price $K_s$. Here $T$ is set to be exactly 30 calendar days (or equivalently, 22 trading days). Comparison of equations (3) and (4) reveals that the payoff for each state is defined to be

$$d_s := \left( \ln \left( \frac{K_s}{S_0} \right) \right)^2$$  \hspace{1cm} (5)

Here we assume an industry standard of zero mean assumption for the short term SPX market returns. We also consider the non-zero mean counterparts, where Treasury-bill yields and the realized mean return of SPX over the most recent thirty trading days are adopted. Results are qualitatively the same.

### 3.3 Modeling State Prices

Having defined the payoff in each state, this section examines on estimating state prices. Breeden and Litzenberger (1978) show that the state price may be modeled as the second derivative of a call or a put option price.

$$\Phi(T, \ldots) = \frac{\partial^2 C(K, T)}{\partial K^2} = \frac{\partial^2 P(K, T)}{\partial K^2}$$

In the risk-neutral framework, the put option price can be defined as\(^8\):

$$P = \int_0^K e^{-rT} (K - S_T)f(S_T) dS_T$$  \hspace{1cm} (6)

where $f(S_T)$ is the risk-neutral probability density function. Note that, in a world where the asset price dynamic is discrete rather than continuous, the above equation can be approximated as

$$P = \sum_s e^{-rT}(K - s)p(s)$$  \hspace{1cm} (7)

where $p(\cdot)$ is the risk-neutral probability of asset price being $s \in (0, K)$.

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\(^7\)Since August 24, 1992, SPX options expire at the open on the third Friday of the contract month. Hence the time to expiration is the number of calendar days remaining less one.

\(^8\)By looking at portfolios construction of butterfly spreads, an alternative derivation is presented in Appendix A.
If we take the partial derivative in the first equation with respect to the strike price $K$, we get

$$\frac{\partial P}{\partial K} = \frac{\partial}{\partial K} \left\{ \int_0^K e^{-rT} (K - S_T) f(S_T) dS_T \right\}$$

$$= e^{-rT} \left\{ (K - K) f(K) + \int_0^K f(S_T) dS_T \right\}$$

$$= e^{-rT} F(K)$$

where $F(\cdot)$ is the risk-neutral distribution function. Taking the second derivative with respect to strike price $K$, we get

$$\frac{\partial^2 P}{\partial K^2} = \frac{\partial}{\partial K} \left\{ e^{-rT} F(K) \right\} = e^{-rT} f(K)$$

Rearranging the above function, it gives:

$$f(K) = e^{rT} \frac{\partial^2 P}{\partial K^2}$$

By examining the put-call parity, it can be shown that for call options, the same relationship is held:

$$f(K) = e^{rT} \frac{\partial^2 C}{\partial K^2}$$

The problem is reduced to estimating the second derivative of option prices with respect to strike price. In this paper, we consider both model-based and model-free approaches to estimate state prices. While they provide similar estimates in a controlled environment, the model-based approach is chosen for its ability to avoid the negative state price issue raising in the data.

In the model-free approach, the state price is computed as the mathematical approximation of the second derivative:

$$\phi_i = e^{rT} \frac{\partial^2 Q_i}{\partial K^2} \approx e^{rT} \frac{(Q_{i-1} - Q_i) - (Q_i - Q_{i+1})}{(\Delta K)^2} = e^{rT} \frac{Q_{i-1} - 2Q_i + Q_{i+1}}{(\Delta K)^2}$$

where $Q_i$ represents a call or a put option price. More precisely, the above equations can be extended using more Taylor series (Eberly, 2008):

$$\phi_i = e^{rT} \frac{\partial^2 Q_i}{\partial K^2} \approx e^{rT} \frac{-Q_{i-2} + 16Q_{i-1} - 30Q_i + 16Q_{i+1} - Q_{i+2}}{12(\Delta K_i)^2}$$

These can be directly estimated with observed SPX option prices. We perform simulations to analyzed the approximation errors of the forward-looking volatility estimation using the model-free approach. We simulate option prices with the Black-
Scholes model using the following parameters. The S&P 500 index level is set to be 995 and the volatility is 30%. Two option maturities, 17 and 45 days are used to interpolate 30-day volatility measure. Strike prices range from 100 to 2000 with strike increment varies between 0.1 and 25. The risk-free rate and dividend yield are 3.08% and 2%, respectively.

<table>
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<th>True σ</th>
<th>CBOE VIX</th>
<th>Model-Free State-Price</th>
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<td>0.29999</td>
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Table 1: Simulation Results of Using Model-Free Approach to Compute State Price. We simulate option prices with the Black-Scholes model using the following parameters. The S&P 500 index level is set to be 995 and the volatility is 30%. Two option maturities, 17 and 45 days are used to interpolate 30-day volatility measure. Strike prices range from 100 to 2000 with strike increment varies between 0.1 and 25. The risk-free rate and dividend yield are 3.08% and 2%, respectively.

An examination of the above table reveals that both CBOE method and state-price approach work on a theoretical ground when the strike interval is small. As strike interval becomes wider, estimation errors are more pronounced for state-price approach.

Although the model-free approach gives reasonable estimates of true volatility with simulated data; however, it fails to incorporate with the real world SPX options data, as shown in three phenomena.

First, option prices for deep OTM options are occasionally equal across a consecutive series of strikes and thus a zero state price is obtained. This is more common for less volatile trading days, where less trades/quotes are observed in each end of strike series.

Second, irrational bids are seen in deep OTM options. For example, an OTM put with lower strike price has higher average of bid-ask spread than another OTM put with higher strike price, where theoretically the former option is strictly less than the latter one. This would return a negative state price.

Third, even when the price of OTM put option is an increasing function of its strike price, it is still possible to obtain a negative state price whenever $P_{i-1} - 2P_{i} + P_{i+1} < 0$. It is worth noting that this does not necessarily lead to an arbitrage opportunity as the price of an option is defined as the average of highest bid and lowest ask.

For the above reasons regarding the failure with the model-free approach, we adopt a model-based approach. In the Black-Scholes framework, Breeden and Litzenberger (1978) show that the state price may be approximated by the so called
delta security, as shown in the equation below.

\[ \Phi(K_i, K_{i+1}) = e^{-rT}(N(d_2(K_i)) - N(d_2(K_{i+1}))) \]  

(10)

where \( K_i < K_{i+1} \) and

\[ N(d_2(K)) = \frac{\ln(S_0/K) + (r - d - \sigma^2/2)T}{\sigma\sqrt{T}} \]  

(11)

where \( d \) is the dividend yield. The key input in equation 11 is the volatility parameter \( \sigma \), which is estimated as the average of the Black-Scholes implied volatilities from two ATM call and put options from two maturities that are closest to a 30-day period (i.e. an average of four implied volatilities). A substantial amount of empirical studies document that ATM Black-Scholes implied volatility is an efficient forecast of subsequent realized volatility.

Latane and Rendleman (1976) study the volatilities implied by call option prices on individual stocks. They show that the implied volatilities predict future volatility better than other predictors based on historical stock price data.

By using the constant proportional dividend yield model in Merton (1973) as an extension on Latane and Rendleman (1976), Chiras and Manaster (1978) find that implied variances are better predictors of future stock return variances than those based on historic stock price information. They develop a trading strategy from informational content of the implied variance and show that it produces abnormally high returns.

Beckers (1981) reach the similar conclusion by extending the analysis to include dividends and optimal weighting schemes of different options on the same stock. Day and Lewis (1988) study the volatility of the stock market around the quarterly expirations of stock index futures contracts and the non-quarterly expirations of stock index options. All call options on the Major Market Index, the New York Stock Exchange Composite Index and S&P 100 Index from 1983 and 1986 are examined. They show that the behavior of implied volatilities reflects increases in the volatility of stock indices around expiration dates.

By investigating implied volatilities of both call and put options on S&P 100 index, Fleming (1998) finds that the implied volatility dominates the historical volatility in terms of ex ante forecasting power. He studies the forecast error from the implied volatility and shows that it is orthogonal to parameters linked to conditional volatility, such as ones found in ARCH specifications. Christensen and Prabhala (1998) make the similar conclusion in the S&P 100 index market.

Similar favorable evidence is found in the currency market. Jorion (1995) examines implied volatilities of short-term ATM options on the German deutsche mark,
the Japanese yen and the Swiss franc. He finds that implied volatility outperforms time-series forecasts, such as a moving average and GARCH estimate, in terms of both the informational content and the predictive power. Collectively, these literatures support the empirical use of the implied volatility as a reasonable input in our state prices.

Since the state price at each strike price no longer depends on the pricing information as seen in (11), we create states, with 0.10 increments, range from half of the lowest SPX level to one and half times the highest SPX level in our sample from 1996 to 2010. This corresponds to a range of 300 to 2400 with 21001 states. Since each state price now represents an interval from \( K_i \) to \( K_{i+1} \), we need to modify the state payoffs to get to the center of the 10 cents interval.

\[
d_s := \left( \ln \left( \frac{K_s + 0.05}{S_0} \right) \right)^2
\]

Summing all products of the state payoff and the state price, we obtain the FIX\(^2\). FIX is then calculated as the squared-root of it. It is worth noting that there is no interpolation of two FIXs from two maturities required in the model-based approach. This is because the key input \( \sigma \) in equation (11) is computed as a 30-day measure. Since FIX is based on the pricing information from ATM options, it is much harder and more costly to be manipulated, comparing to VIX.

4 Hypotheses and Results

In the previous section, we present the methodology used to construct FIX. Since FIX is an application of the state-preference pricing approach, it seems intuitive to firstly test whether it qualifies as a volatility estimator. That is, what is the connection between FIX and the current market volatility barometer VIX? We then study whether FIX shares any feature of VIX that has been documented in the literature, such as the negative correlation with the market return and the asymmetric response to market falls from upswings. The next question to be tested is whether FIX developed using state-preference algorithm provides better predictions of future stock market volatilities than VIX and other estimators.

4.1 FIX and VIX: Historical Behaviors

In this section, we start with examining the historical behaviors of FIX and VIX from January 4, 1996 to October 29, 2010. Table 2 reports the summary statistics on the levels and logarithmic returns of FIX, the two volatility indices VIX and VXO, and the corresponding subsequent 30-day realized volatilities in SPX market RVol\(^{SPX}\).
Logarithmic returns are defined as, for example, $\Delta \text{FIX}_t = \ln(\text{FIX}_t/\text{FIX}_{t-1})$. Each series has 3,731 daily observations and all entries are represented in percentage volatility points.

We follow the industry standard for the zero mean assumption in realized variance calculation. The *ex post* monthly realized volatility is estimated as:

$$\text{RVol}_{t,t+30} = 100 \times \sqrt{\frac{365}{30} \sum_{i=1}^{22} \left( \ln \left( \frac{S_{t+i}}{S_{t+i-1}} \right) \right)^2} \quad (12)$$

As shown in the table, the sample mean, median and maximum of VXO are the highest, which confirm with the artificial upward bias from the trading day conversion discussed in the previous section. Comparing VIX with the realized volatility measures in SPX market, we find that VIX is approximately 4 percentage points higher on average and this is in line with the finding in Carr and Wu (2006). FIX is approximately 2 percentage points higher than the realized volatility measures on average.

The maximum of VIX and VXO in our sample occur on November 20, 2008 at 80.86 and 87.24, respectively. On that date, FIX is at its second highest level of 75.109 (and the highest level occur on October 10, 2009 at 81.81). Although not included in the table, the top 10 highest forward looking realized volatilities occur around the beginning of October, 2008. During that period, both VIX and FIX are around 45. That implies that even though VIX and FIX over-estimate the true monthly realized volatility on average, they fail to capture them in the extreme market conditions. This is equivalent as saying those extreme volatilities are not foreseen by traders in SPX options market.

The sample standard deviations of the realized volatilities are higher than all volatility indices. This is consistent with findings in Carr and Wu (2006). Moderate positive skewness and serious kurtosis are shown in all volatility indices and more so in realized volatilities. Similar patterns are observed in the daily logarithmic returns.

---

\(^9\)Note that this formulation adopts the calendar day annualization. Alternatively, one can use the trading day annualization, where the summation in equation (12) is multiplied by 252/22. As discussed in Carr and Wu (2009), volatility is a continuous process. The underlying assumption in the calendar day annualization is that the variance over weekends should be 3 times of the average weekday variance. However, by investigating the distribution of SPX returns, Carr and Wu (2009) show that the variance over weekend is only 1.67 times bigger. On the other hand, the trading day annualization may overestimate the weekend variance as the trading day returns contain significant negative serial correlation up to a 2-day lag. Since there is no commonly agreed standard around whether the annualization, we tabulate results for both estimates. We denote the *ex post* monthly realized volatility using trading day conversion as $\text{RVol}_{t,t+22}$. 

16
Moments | FIX | VIX | VXO | RVol_{30} | RVol_{22} | ΔFIX | ΔVIX | ΔVXO | ΔRVol_{30} | ΔRVol_{22}  
--- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | 
Mean   | 20.44 | 22.21 | 23.14 | 18.71 | 18.15 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 
Median  | 19.42 | 21.01 | 22.25 | 16.43 | 15.94 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 
Maximum | 84.44 | 80.86 | 87.24 | 90.57 | 87.88 | 0.54 | 0.50 | 0.53 | 0.66 | 0.66 | 
Minimum | 8.31  | 9.89  | 0.00  | 5.86  | 5.68  | -0.55 | -0.35 | -0.38 | -0.39 | -0.39 | 
Std. Dev. | 8.33  | 8.73  | 9.57  | 10.69 | 10.38 | 0.07 | 0.06 | 0.07 | 0.06 | 0.06 | 
Skewness | 2.02  | 1.91  | 1.70  | 2.74  | 2.74  | 0.31 | 0.50 | 0.47 | 0.74 | 0.74 | 
Auto    | 0.97  | 0.98  | 0.97  | 0.99  | 0.99  | -0.17 | -0.09 | -0.14 | -0.01 | -0.01 | 

Table 2: Summary Statistics of FIX, VIX, VXO and Realized SPX Return Volatilities. This table reports the sample average (Mean), median, maximum, minimum, standard deviation (Std. Dev.), skewness, excess kurtosis and first-order autocorrelation (Auto) on the levels and logarithmic returns of FIX, VIX, VXO, the 30-day realized volatility with calendar days convention (RVol_{30}) and with trading days convention (RVol_{22}). The logarithmic return is defined as, for example, \(ΔFIX_t = \ln(\text{FIX}_t/\text{FIX}_{t-1})\). Each series has 3,731 daily observations from January 4, 1996 to October 29, 2010. All entries are represented in percentage volatility points.
Table 3 reports the cross-correlations between the three volatility indices and the forward looking realized volatilities. All three volatility indices are positively correlated with the subsequent realized volatilities; however, these correlations become negligible in the logarithmic returns. This is in line with findings in the previous literature (e.g., Carr and Wu (2006)). Although based on completely different methodologies and principles, FIX is highly positively correlated with VIX at 99.1%. It is worth noting that FIX is marginally more correlated with VIX than VXO with VIX, which is 98.9%. For the logarithmic returns, the correlation between ∆FIX and ∆VIX is 89.9%. This implies that FIX can be an alternative of VIX for its function as a volatility predictor in the market. This is further discussed in the next section.

The positive strong correlation between FIX and VIX is better illustrated in Figure 1. It shows daily closing levels of FIX, VIX, VXO and the scaled S&P 500 index from January 4, 1996 through October 29, 2010. Reading from top to bottom, the first trajectory is the scaled S&P 500 index. The next trajectories are VXO, VIX and FIX. It is difficult to distinguish them in the diagram due to the remarkable high correlations.

Apart from the high correlations among these volatility indices, another interesting phenomenon worth mentioning is their negative correlations with SPX. That is, these volatility indices spike upward when there is a crash in the market. This negative relationship is well documented in the literature (e.g., Whaley (2009)). This is more pronounced during the Global Financial Crisis from 2008, towards the right end of the figure.

10 This scaled S&P 500 index is obtained by dividing the original index by 15.
<table>
<thead>
<tr>
<th>Correlation</th>
<th>FIX</th>
<th>VIX</th>
<th>VXO</th>
<th>RVol(_{30})</th>
<th>RVol(_{22})</th>
<th>∆FIX</th>
<th>∆VIX</th>
<th>∆VXO</th>
<th>∆RVol(_{30})</th>
<th>∆RVol(_{22})</th>
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</thead>
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<tr>
<td>FIX</td>
<td>1.000</td>
<td>0.990</td>
<td>0.987</td>
<td>0.775</td>
<td>0.775</td>
<td>-0.103</td>
<td>-0.084</td>
<td>-0.087</td>
<td>-0.095</td>
<td>-0.095</td>
</tr>
<tr>
<td>VIX</td>
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<td>0.987</td>
<td>0.761</td>
<td>0.761</td>
<td>-0.083</td>
<td>-0.087</td>
<td>-0.084</td>
<td>-0.092</td>
<td>-0.092</td>
</tr>
<tr>
<td>VXO</td>
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<td>0.987</td>
<td>1.000</td>
<td>0.770</td>
<td>0.770</td>
<td>-0.084</td>
<td>-0.081</td>
<td>-0.091</td>
<td>-0.096</td>
<td>-0.096</td>
</tr>
<tr>
<td>RVol(_{30})</td>
<td>0.775</td>
<td>0.761</td>
<td>0.770</td>
<td>1.000</td>
<td>1.000</td>
<td>0.062</td>
<td>0.071</td>
<td>0.069</td>
<td>-0.046</td>
<td>-0.046</td>
</tr>
<tr>
<td>RVol(_{22})</td>
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<td>0.761</td>
<td>0.770</td>
<td>1.000</td>
<td>1.000</td>
<td>0.062</td>
<td>0.071</td>
<td>0.069</td>
<td>-0.046</td>
<td>-0.046</td>
</tr>
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<td>∆FIX</td>
<td>-0.103</td>
<td>-0.083</td>
<td>-0.084</td>
<td>0.062</td>
<td>0.062</td>
<td>1.000</td>
<td>0.882</td>
<td>0.878</td>
<td>-0.010</td>
<td>-0.010</td>
</tr>
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<td>-0.087</td>
<td>-0.081</td>
<td>0.071</td>
<td>0.071</td>
<td>0.882</td>
<td>1.000</td>
<td>0.911</td>
<td>-0.016</td>
<td>-0.016</td>
</tr>
<tr>
<td>∆VXO</td>
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<td>-0.084</td>
<td>-0.091</td>
<td>0.069</td>
<td>0.069</td>
<td>0.878</td>
<td>0.911</td>
<td>1.000</td>
<td>-0.009</td>
<td>-0.009</td>
</tr>
<tr>
<td>∆RVol(_{30})</td>
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<td>-0.092</td>
<td>-0.096</td>
<td>-0.046</td>
<td>-0.046</td>
<td>-0.010</td>
<td>-0.016</td>
<td>-0.009</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>∆RVol(_{22})</td>
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<td>-0.092</td>
<td>-0.096</td>
<td>-0.046</td>
<td>-0.046</td>
<td>-0.010</td>
<td>-0.016</td>
<td>-0.009</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 3: Cross-Correlations of FIX, VIX, VXO and 30-day Forward Looking Realized SPX Return Volatilities. This table reports cross-correlations on the levels and logarithmic returns of FIX, VIX, VXO, the 30-day realized volatility with calendar days convention (RVol\(_{30}\)) and with business days convention (RVol\(_{22}\)). The logarithmic return is defined as, for example, ∆FIX\(_{t}\) = ln(FIX\(_{t}\)/FIX\(_{t-1}\)). Each series has 3,731 daily observations from January 4, 1996 to October 29, 2010.
Figure 1: Daily Closing Levels of FIX, VIX, VXO andScaled S&P 500 Index from January 4, 1996 to October 29, 2010.
Whaley (2009) documents that the change in VIX rises at a higher absolute rate of change when there is a market fall than an upswing. To test the proposition whether FIX shares this feature of VIX, we regress the daily rate of change of FIX, $RFIX_t = \ln(FIX_t/FIX_{t-1})$, against the rate of change of S&P 500 index, $RSPX_t = \ln(SPX_t/SPX_{t-1})$ and the rate of change of S&P 500 index conditional on whether it is a negative return or 0 otherwise i.e. $RSPX_t^- = \min(0, RSPX_t)$. We also repeat the test for VIX as the benchmark result for our study. Our regressions are in the following forms,

$$RFIX_t = \alpha_0 + \alpha_1 RSPX_t + \alpha_2 RSPX_t^- + \epsilon_1$$
$$RVIX_t = \beta_0 + \beta_1 RSPX_t + \beta_2 RSPX_t^- + \epsilon_2$$

(13)

If the proposition is true, we expect $\alpha_0$ to be not statistically significant different from 0, while both $\alpha_1$ and $\alpha_2$ should be significantly less than 0. Table 4 reports the regression results. Since we are examining the daily changes, there are 3,730 observations from January 5, 1996 to October 29, 2010 in each series. Intercept $\alpha_0$ is not significant different from 0 at 10% level. Interestingly, we find that $\beta_0$ is significantly different from 0 at 1% level, which somehow contradicts with the result in Whaley (2009); however, at the level of $-0.0024$, this difference may not be economically significant.

As expected, all slope coefficients are significantly less than 0. This implies that not only there is an inverse relation between changes in FIX/VIX and that in SPX, but also the response to different swings in the market is asymmetric. This is consistent with findings in Whaley (2009).

To better understand these results, one may suppose an increase in SPX by 100 basis points, this will result

$$RFIX_t = -3.57 \times (0.01) = -3.57\%$$
$$RVIX_t = -3.02 \times (0.01) = -3.02\%$$

On the contrary, a decrease in SPX by 100 basis points will result

$$RFIX_t = -3.57 \times (-0.01) - 0.48 \times (-0.01) = 3.95\%$$
$$RVIX_t = -3.02 \times (-0.01) - 0.78 \times (-0.01) = 3.80\%$$

The above example illustrates that the relation between movements in FIX/VIX and movements in SPX is asymmetric. It is worth noting that the interpretation is limited to the correlation, rather than any causality in between (Whaley, 2009).

As a conclusion, we show that as an application of the state-preference pricing approach, FIX behaves like the current market volatility barometer VIX and thus it qualifies as a volatility estimate.
### Table 4: Regression Results of RFIX<sub>t</sub> and RVIX<sub>t</sub> against the rate of change of S&P 500 index, RSPX<sub>t</sub> = ln(SPX<sub>t</sub>/SPX<sub>t−1</sub>) and the rate of change of S&P 500 index conditional on whether it is a negative return or 0 otherwise i.e. RSPX<sub>t</sub><sup>-</sup> = min(0, RSPX<sub>t</sub>). There are 3,730 daily observations from January 5, 1996 to October 29, 2010 in each series. Regressions equations are presented in equation (13). The top half of the table reports results with the dependent variable RFIX and the bottom half of the table reports that with RVIX. Both volatility estimates calculate the realized volatility using 22 daily logarithmic returns.

<table>
<thead>
<tr>
<th></th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistics</th>
<th>Prob.</th>
<th>Adj. R&lt;sup&gt;2&lt;/sup&gt;</th>
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<tbody>
<tr>
<td>RFIX&lt;sub&gt;t&lt;/sub&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RSPX&lt;sub&gt;t&lt;/sub&gt;</td>
<td>-3.5732</td>
<td>0.1068</td>
<td>-33.4548</td>
<td>0.0000</td>
<td>0.5100</td>
</tr>
<tr>
<td>RSPX&lt;sub&gt;t&lt;/sub&gt;&lt;sup&gt;-&lt;/sup&gt;</td>
<td>-0.4764</td>
<td>0.1687</td>
<td>-2.8231</td>
<td>0.0048</td>
<td></td>
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<tr>
<td>Intercept</td>
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<td>0.0011</td>
<td>-1.2175</td>
<td>0.2235</td>
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<tr>
<td>RVIX&lt;sub&gt;t&lt;/sub&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>RSPX&lt;sub&gt;t&lt;/sub&gt;</td>
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<td>RSPX&lt;sub&gt;t&lt;/sub&gt;&lt;sup&gt;-&lt;/sup&gt;</td>
<td>-0.7765</td>
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<td></td>
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<tr>
<td>Intercept</td>
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<td>0.0009</td>
<td>-3.1278</td>
<td>0.0018</td>
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</tr>
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</table>
4.2 Predictability Comparison of FIX, VIX and VXO

In the previous section, we show that although based on completely different methods and principals, FIX is highly correlated with VIX in terms of both levels and daily logarithmic changes. FIX shares with many features of VIX, such as the negative correlation with market returns and asymmetric movements from a market downswing or upswing. We also show that FIX is less data dependent, easier to compute and more difficult for a manipulation, due to the fact that FIX is not affected by an artificial bid in deep OTM options even when there is no trading exchange hands.

As outlined in CBOE (2009), the primary goal of VIX is to measure the 30-day expected volatility of the S&P 500 index. Therefore it seems intuitive to test the volatility predictability of FIX against VIX.

As discussed in section 3.3, FIX is constructed in the state-preference pricing framework with the key input in the state prices as the average implied volatilities of ATM call and put options from two maturities around 30-day. Thus our hypothesis for testing the volatility predictability of FIX against VIX can be inferred as testing the predictability of information from ATM implied volatilities against information from all OTM options.

That is, do previous literature findings (e.g., Latane and Rendleman (1976); Beckers (1981); Day and Lewis (1988); Lamoureux and Lastrapes (1993); Fleming (1998); Gwilym and Buckle (1999)) hold with SPX options data from 1996 to 2010? Does the inclusion of OTM options in VIX construction add extra information or noise in predicting future market volatility?

Since FIX is highly correlated with VIX, we cannot include them in the same regression to see which one is more statistically significant. To test this proposition, we perform a standard test to regress the \textit{ex post} monthly realized volatility on FIX and VIX, respectively. In addition, even though VXO is based on the information from ATM implied volatilities of OEX options, we repeat the test for VXO since SPX market is highly correlated with OEX market. Our regressions are in the following forms,

\begin{align*}
R_{Vol,t+30} &= \alpha_0 + \alpha_1 \text{FIX}_t + \epsilon_1 \\
R_{Vol,t+30} &= \beta_0 + \beta_1 \text{VIX}_t + \epsilon_2 \\
R_{Vol,t+30} &= \gamma_0 + \gamma_1 \text{VXO}_t + \epsilon_3
\end{align*}

If FIX is a better predictor than VIX and VXO, we expect \( \alpha_0 \) to be not significantly different from 0 and \( \alpha_1 \) to be not significantly different from 1. In addition, the first equation with FIX is expected to have the highest adjusted R-squared.

The top half of Table 5 reports results of the above regressions. There are 3,731 daily observations from January 4, 1996 to October 29, 2010 in each series. The
<table>
<thead>
<tr>
<th>RVol_{30}</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistics</th>
<th>Prob.</th>
<th>Adj. R^2</th>
<th>Wald Test</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>-1.6562</td>
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<tr>
<td>VXO</td>
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<td>RVol_{22}</td>
<td></td>
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</tr>
<tr>
<td>VXO</td>
<td>0.8302</td>
<td>0.0546</td>
<td>15.2183</td>
<td>0.0000</td>
<td>0.0019</td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>-1.0740</td>
<td>1.1077</td>
<td>-0.9696</td>
<td>0.3323</td>
<td>0.5893</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Regression Results of FIX, VIX, VXO against 30-day Forward Looking Realized SPX Return Volatilities. There are 3,731 daily observations from January 4, 1996 to October 29, 2010 in each series. The heteroskedasticity-consistent standard errors and covariance matrix is computed according to Newey and West (1987) with the lag truncation of 8. Regressions equations are presented in equation (14). The top half of the table reports results with the dependent variable RVol_{30}, which is the subsequent realized volatility in calendar days convention. The bottom half of the table reports that with RVol_{22}, which is in business days convention. Both volatility estimates calculate the realized volatility using 22 daily logarithmic returns. The last column on the right reports the F-statistics in the Wald Test, which is to test whether the slope coefficient is significantly different from 1. The lower the number, the higher the chance to be significantly different from 1.
covariance matrix is computed according to Newey and West (1987) with the lag truncation of 8 to correct for any potential serial correlations in the error terms. When FIX is the explanatory variable, it gives the highest adjusted R-squared with 0.6250. This implies that using FIX alone provides more explanatory power in predicting future realized volatility. It is quite surprising that the second highest adjusted R-squared comes from the regression with VXO at 0.6134.

All slope coefficients are significantly different from 0 and intercept coefficients of FIX and VXO are not significant at 5% level. It is worth noting that, in order to be qualified as a good proxy for future realized volatility, we are more interested in testing whether those slope coefficients are different from 1. To test that, we perform Wald Tests and report the F-statistics in the last column in Table 5. It is quite pronounced that the test statistics with FIX returns the highest number, 0.7520, which implies that the null hypothesis of $\alpha_1 = 1$ fails to be rejected.

We also perform the test with the subsequent realized volatilities in business days convention. Quantitatively speaking, results are not much different and hence similar discussions can be made.

Based on these findings in the above regressions, we conclude that FIX is a better predictor for future realized volatility, than VIX and VXO. In addition, the inclusion of volatility information from OTM options does not add extra power in predicting future market realized volatility.

### 4.3 Predictability Comparison of FIX and other Volatility Forecast Estimators

In this section, we further compare the volatility predictability of FIX with the squared root of GARCH(1,1) variance and other model-free implied volatility estimates provided by Investment Banks.

Carr and Wu (2006) study the predictability of realized variance by using VIX$^2$ and GARCH(1,1) variance. They show that GARCH variance does not provide much extra information over VIX when both are included in the regression. In addition, VIX$^2$ provides a better explanatory power (with approximately 10% higher adjusted R-squared) as the only predictor in the regression. Here we replicate the study on different data sample. To estimate GARCH(1,1) variance, we use an AR(1) assumption on the S&P 500 index return process. GARCH(1,1) volatility is obtained using the following formula:

$$GARCH(1,1)_{vol} = 100 \times \sqrt{GARCH(1,1)_{var} \times \frac{365}{30}}$$

We obtain the model-free implied variance estimates provided by Bank of Amer-
ica Merrill Lynch (BAML) and J.P. Morgan (JPM). We then take the squared-root to obtain the volatility counterparts. Regression formulas are similar to the ones presented in equation (14), where the explanatory variables are changed accordingly.

\[
\begin{align*}
RV_{t,t+30} &= \alpha_0 + \alpha_1 \text{GARCH}(1,1)_t + \epsilon_1 \\
RV_{t,t+30} &= \beta_0 + \beta_1 \text{BAML}_t + \epsilon_2 \\
RV_{t,t+30} &= \gamma_0 + \gamma_1 \text{JPM}_t + \epsilon_3
\end{align*}
\]

Results are presented in Table 6. We also include the regression result of FIX from previous table for comparison. Model-free implied volatility estimates provide better prediction about future realized volatility than GARCH(1,1) volatility, with higher adjusted R-squared and less biased slope coefficients (as shown by the larger F-statistics in the last column). Overall, it is quite clear that FIX outperforms the other forecast estimators.

We conclude that FIX, constructed in the state-preference pricing approach with Black-Scholes analytic second derivatives, is an efficient and more superior forecast of subsequent realized volatility.

5 Conclusion

As an application of the state-preference pricing approach with Black-Scholes analytic second derivatives as state prices, we develop a forward-looking volatility index FIX as a proxy for the S&P 500 index return realized volatility in the next 30-day. FIX is less data dependent, easier to compute and more difficult for a manipulation, due to the fact that FIX is not affected by an artificial bid in deep OTM options even when there is no trading exchange hands.

We show that despite being based on completely different methods and principals, FIX is highly correlated with VIX in terms of both levels and daily logarithmic changes. FIX shares with many features of VIX, such as the negative correlation with market returns and asymmetric movements from a market downswing or up-swing.

By comparing the predictability of future monthly realized volatility among FIX, VIX, VXO, GARCH(1,1) volatility and two other volatility estimates provided by Bank of America Merrill Lynch and J.P. Morgan, we show that FIX outperforms all other forecasts. Using FIX as the sole explanatory variable, the regression model provides the highest explanatory power (the highest adjusted R-squared). We fail to reject the null hypothesis and the coefficient of FIX is not significantly different.

\footnote{These variance estimates are obtained from replicating variance swaps; however, exact methodologies used to compute each model-free variance are not provided.}
<table>
<thead>
<tr>
<th>RVol_{30}</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistics</th>
<th>Prob.</th>
<th>Adj. R^2</th>
<th>Wald Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIX</td>
<td>0.9978</td>
<td>0.0660</td>
<td>15.1230</td>
<td>0.0000</td>
<td>0.9728</td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>-1.6155</td>
<td>1.1784</td>
<td>-1.3710</td>
<td>0.1705</td>
<td>0.6043</td>
<td></td>
</tr>
<tr>
<td>GARCH(1,1)</td>
<td>2.6767</td>
<td>0.2583</td>
<td>10.3626</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>3.4280</td>
<td>1.2960</td>
<td>2.6451</td>
<td>0.0082</td>
<td>0.4522</td>
<td></td>
</tr>
<tr>
<td>BAML</td>
<td>0.9287</td>
<td>0.0584</td>
<td>15.9078</td>
<td>0.0000</td>
<td>0.2221</td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>-1.5305</td>
<td>1.1138</td>
<td>-1.3741</td>
<td>0.1695</td>
<td>0.5941</td>
<td></td>
</tr>
<tr>
<td>JPMorgan</td>
<td>0.9478</td>
<td>0.0639</td>
<td>14.8249</td>
<td>0.0000</td>
<td>0.4141</td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>-1.8253</td>
<td>1.2175</td>
<td>-1.4992</td>
<td>0.1339</td>
<td>0.5661</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Regression Results of GARCH(1,1) Volatility, Model-Free Volatility Estimates from Bank of America Merrill Lynch and J.P. Morgan against 30-day Forward Looking Realized SPX Return Volatilities. There are 3,623 daily observations from January 4, 1996 to May 27, 2010 in each series. The heteroskedasticity-consistent standard errors and covariance matrix is computed according to Newey and West (1987) with the lag truncation of 8. Regressions equations are presented in equation 15. Forward looking realized volatility estimate RVol_{30} is computed based on 22 daily logarithmic returns in the calendar days convention. The last column on the right reports the F-statistics in the Wald Test, which is to test whether the slope coefficient is significantly different from 1. The lower the number, the higher the chance to be significantly different from 1.
from 1. In addition, the intercept term is not significant at 5% level. In other words, FIX is a better estimator of future realized volatility in S&P 500 market.

The key input in the Black-Scholes analytic state prices is computed as an average of implied volatilities of four ATM call and put options. From that perspective, our study reinforce the findings in previous literature that ATM Black-Scholes implied volatility is an efficient forecast of subsequent realized volatility. Since VIX originates from a model-free variance estimate extracted directly from option prices, we deduce that the inclusion of volatility information from OTM options in VIX does not add extra explanatory power in predicting future market volatility.

Appendix

DDKZ’s Fair Value of Future Variance and the New VIX Formula

The methodology of the new VIX originates from the seminal paper by Demeterfi et al. (1999). The principal of their approach is to show how the dynamic hedging of a log contract captures realized volatility. Equation (26) in DDKZ (1999) is the core formula in the fair value of future variance:

\[
\sigma^2_{DDKZ} = \frac{2}{T} \left( rT - \left( \frac{S_0}{S_*} e^{rT} - 1 \right) - \ln \left( \frac{S_*}{S_0} \right) \right) + e^{rT} \int_0^{S_*} \frac{P(K)}{K^2} dK + e^{rT} \int_{S_*}^{\infty} \frac{C(K)}{K^2} dK
\]

(10)

where \( S_* \) is the approximate at-the-money forward stock level that marks the boundary between liquid puts and liquid calls. CBOE defines \( S_* := K_0 \), where \( K_0 \) (same as in equation (1)) is the first strike below the forward index level \( F = S_0 e^{rT} \). For the derivation here, we stick with \( S_* \). To show how DDKZ arrives in this equation, it should be understood that the theoretical definition of realized variance for a given price history is

\[
V = \frac{1}{T} \int_0^T \sigma^2(t, \ldots) dt
\]

If we think about pricing a variance swap, the value of such swap can be written as \( F = E(e^{-rT}(V - K)) \). That is, for a zero initial value, we obtain the strike \( K_{var} \):

\[
K_{var} = E(V) = \frac{1}{T} E \left( \int_0^T \sigma^2(t, \ldots) dt \right)
\]

They assume that the future underlyer evolution is diffusive (i.e. no jumps allowed):
\[
\frac{dS_t}{S_t} = \mu(t, \ldots)dt + \sigma(t, \ldots)dZ_t
\]

Ito's lemma \[\Rightarrow\] 
\[d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t\]

\[\Rightarrow \frac{dS_t}{S_t} - d(\ln S_t) = \frac{1}{2}\sigma^2 dt\]

or \[\sigma^2 dt = 2 \left(\frac{dS_t}{S_t} - d(\ln S_t)\right)\]

\[\therefore E(V) = K_{\text{var}} = 2TE\left(\int_0^T dS_t - \int_0^T d(\ln S_t)\right)\]

\[= 2T E\left(\int_0^T \frac{dS_t}{S_t} - \frac{2}{T} E\left(\frac{S_T}{S_0}\right)\right)\]

They then further assume that this is in the risk-neutral framework where the expected asset return is \(r_T\) for time period \(T\). This simplifies the above term \(A\) and \(B\) as:

\[A = E\left[\int_0^T (r dt + \sigma(t, \ldots) dZ_t)\right] = rT \quad Z_t \sim N(0, t)\]

\[B = E\left(\ln \frac{S_T}{S_0}\right) = E\left(\ln \frac{S_T}{S_*}\right) + \ln \frac{S_*}{S_0}\]

where \(S_*\) is as defined above. The next question is to value \(E(\ln(S_T/S_*))\). Rather than deriving it directly, suppose we buy a portfolio of options, \(\Pi\), spanning all strikes \(K \in (0, \infty)\) with expiration \(T\) and weighted inversely proportional to \(K^2\), we have

\[\Pi = \begin{cases} \int_0^{S_*} \frac{1}{K^2} \max(K - S_T, 0) dK + \int_{S_*}^\infty \frac{1}{K^2} \max(S_T - K, 0) dK & \text{OTM puts} \\ \int_{S_T}^\infty \frac{1}{K^2} \max(S_T - K, 0) dK & \text{OTM calls} \end{cases}\]

\[= \begin{cases} \int_{S_T}^{S_*} \frac{1}{K^2} (K - S_T) dK & \text{if } S_T < S_* \\ \int_{S_*}^\infty \frac{1}{K^2} (S_T - K) dK & \text{if } S_T \geq S_* \end{cases}\]

\[= -1 - \ln S_T + \frac{S_T}{S_*} + \ln S_*\]

\[= \frac{S_T - S_*}{S_*} - \ln \frac{S_T}{S_*}\]

\[\therefore E\left(\ln \frac{S_T}{S_*}\right) = E\left(\frac{S_T - S_*}{S_*} - \Pi\right)\]
\[ K_{\text{var}} = \frac{2}{T}(rT) - \frac{2}{T} \left[ E \left( \frac{S_T - S_s}{S_s} \right) - \Pi + \ln \frac{S_s}{S_0} \right] \]

\[ = \frac{2}{T} \left[ rT - E \left( \frac{S_T}{S_s} - 1 \right) + E \left( \int_0^{S_s} \frac{1}{K^2} \max(K - S_T, 0) dK \right) - \ln \frac{S_s}{S_0} \right] \]

\[ = \frac{2}{T} \left[ rT - \left( \frac{S_0 e^{rT}}{S_s} - 1 \right) - \ln \frac{S_s}{S_0} + \left( e^{rT} \int_0^{S_s} \frac{P(K)}{K^2} dK + e^{rT} \int_{S_s}^{\infty} \frac{C(K)}{K^2} dK \right) \right] \]

We now show how to get the VIX formula from DDKZ’s fair value of future variance denoted as \( K_{\text{var}} \). It is quite clear that if one applies discretization and truncation approximation, the numerical integration may be approximated by the summation series. For part b from the above formula, we have:

\[ b = \frac{2 e^{rT}}{T} \left( \int_0^{K_L} \frac{P(K)}{K^2} dK + \int_{K_L}^{K_H} \frac{C(K)}{K^2} dK \right) \approx \frac{2 e^{rT}}{T} \left( \int_0^{K_L} \frac{P(K)}{K^2} dK + \int_{K_L}^{K_H} \frac{C(K)}{K^2} dK \right) \]

\[ \approx \frac{2}{T} \left( \sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) \right) \]

where \( K_L \) and \( K_H \) denotes the lowest and highest strikes in the option strips, respectively. For part a, we apply that \( F = S_0 e^{rT} \) and have:

\[ a = \frac{2}{T} \left( \ln (e^{rT}) - \left( \frac{S_0 e^{rT}}{S_s} - 1 \right) - \ln(S_s) + \ln(S_0) \right) \]

\[ = \frac{2}{T} \left( \ln \left( \frac{F}{S_s} \right) - \left( \frac{F}{S_s} - 1 \right) \right) \]

where \( F = S_0 e^{rT} \)

\[ \approx \frac{2}{T} \left( \left( \frac{F}{S_s} - 1 \right) - \frac{1}{2} \left( \frac{F}{S_s} - 1 \right)^2 - \left( \frac{F}{S_s} - 1 \right) \right) \]

\[ = -\frac{1}{T} \left( \frac{F}{S_s} - 1 \right)^2 \]

The approximation in the second last line comes from applying the Taylor series expansion of the logarithm function and ignoring terms higher than the second order. Combine the simplified results in part a and b, we obtain the new VIX formula:

\[ V \approx \frac{2}{T} \left( \sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) \right) - \frac{1}{T} \left( \frac{F}{S_s} - 1 \right)^2 \]

\[ ^{12} \text{A few maths tricks given here are first applied in Jiang and Tian (2007).} \]
State Prices in Breeden and Litzenberger (1978)

Breeden and Litzenberger (1978) show that the state price may be modeled as the second derivative of a call or a put option price.

$$\frac{\partial^2 C(K,T)}{\partial K^2} = \frac{\partial^2 P(K,T)}{\partial K^2}$$

In section 3.3 we present a mathematical derivation of this state price. Here we present a more intuitive approach by constructing a butterfly spread to long one call with strike \(M - \Delta M\), long one call with strike \(M + \Delta M\) and short two calls with strike \(M\) (Barraclough, 2007). At maturity \(T\), the payoff of this portfolio is illustrated in the following table.

<table>
<thead>
<tr>
<th>(S_T)</th>
<th>(M - \Delta M)</th>
<th>(M - \Delta M &lt; S_T &lt; M)</th>
<th>(M &lt; S_T &lt; M + \Delta M)</th>
<th>(M + \Delta M &lt; S_T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long 1 call with (M - \Delta M)</td>
<td>0</td>
<td>(S_T - (M - \Delta M))</td>
<td>(S_T - (M - \Delta M))</td>
<td>(S_T - (M - \Delta M))</td>
</tr>
<tr>
<td>Short 2 calls with (M)</td>
<td>0</td>
<td>0</td>
<td>(-2(S_T - M))</td>
<td>(-2(S_T - M))</td>
</tr>
<tr>
<td>Long 1 call with (M + \Delta M)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(S_T - (M + \Delta M))</td>
</tr>
<tr>
<td>Total at (t = T)</td>
<td>0</td>
<td>(\Delta M + (S_T - M))</td>
<td>(\Delta M - (S_T - M))</td>
<td>0</td>
</tr>
</tbody>
</table>

That is, the payoff is exactly \(\Delta M\) if \(S_T = M\) at maturity. Thus the cost of butterfly spread that produces a payment of $1 if the future state is \(S_T = M\) is:

$$P(M; \Delta M) = \frac{C(M - \Delta M, T) - 2C(M, T) + C(M + \Delta M, T)}{\Delta M}$$

If we divide the above by the step size \(\Delta M\) and in the limit as \(\Delta M \to 0\), we obtain:

$$\lim_{{\Delta M \to 0}} \frac{P(M; \Delta M)}{\Delta M} = \lim_{{\Delta M \to 0}} \frac{C(M - \Delta M, T) - 2C(M, T) + C(M + \Delta M, T)}{\Delta M^2} = \frac{\partial^2 C(K, T)}{\partial K^2} \bigg|_{K=M}$$

Thus the price of a security \(f\) with payoff \(d_M^f\) at some future state \(M\) is

$$P^f = \int_M d_M^f P(M; \Delta M) = \int_M d_M^f \left( \frac{\partial^2 C(K, T)}{\partial K^2} \bigg|_{K=M} \right) dM$$

References


